Probability Refresher

CMPUT 397
Fall 2020

Reminders: Sept 4, 2020

- Clarification on how we use eClass, Slido and Zoom chat
  - eClass for longer questions
  - Slido for Participation Questions (short questions, that I address in class, related to the material you completed for that module)
  - Zoom chat for questions asked during class
- Practice Quiz and Participation Question **due Tuesday** (participation marks)
- Graded Notebook **due on Friday**

Modeling random outcomes

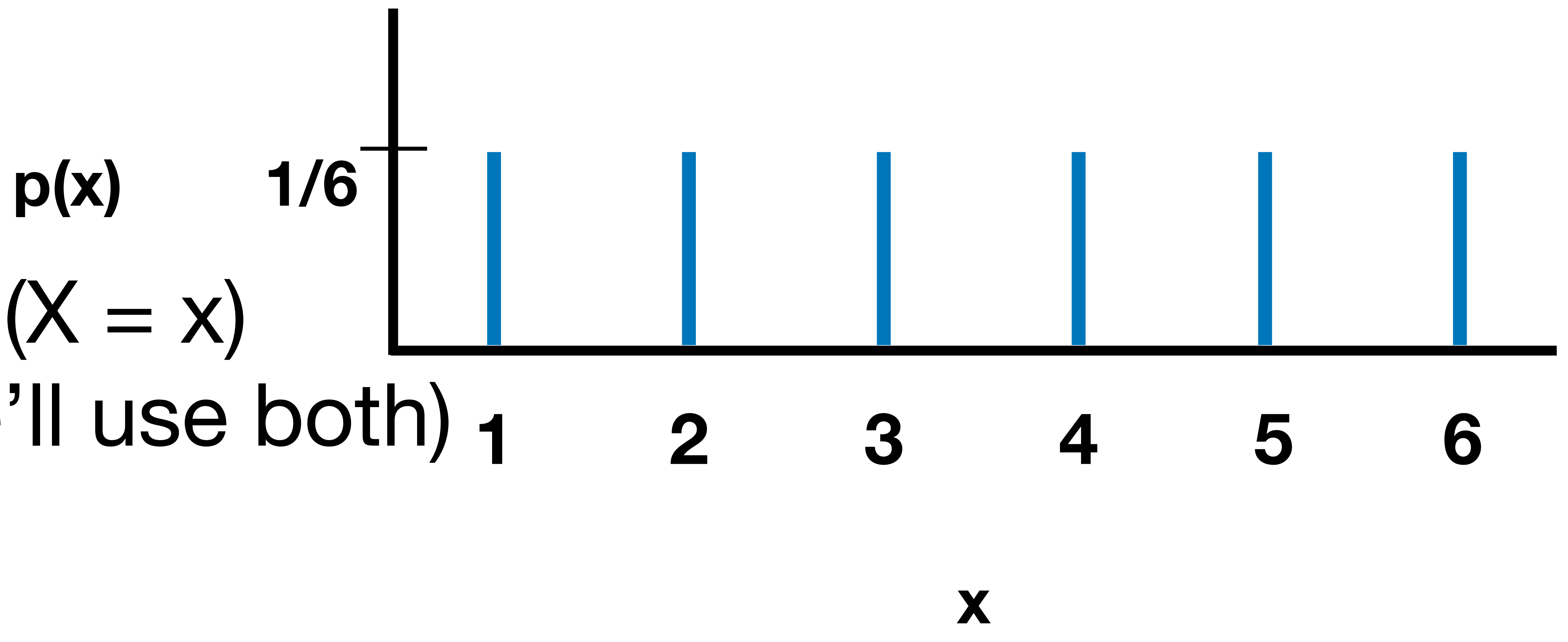
- Imagine you would like to model stochastic outcomes (or dynamics), such as the outcome of a dice role
- We need to define:
  - the possible set of outcomes (e.g., 1, 2, 3, 4, 5, 6)
  - the probability of each outcome (e.g., 1/6 for each)

Random Variable

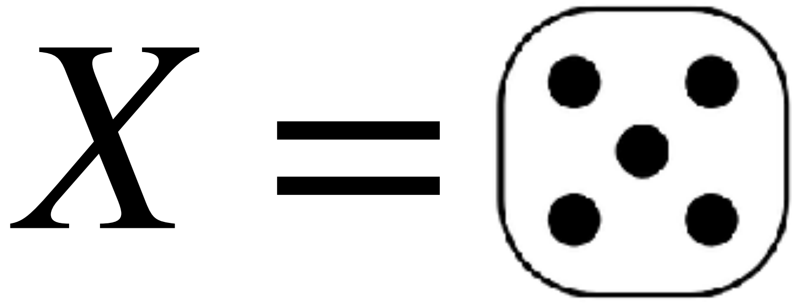
- (Informally) A random variable X is a variable with stochastic values
- Depends on underlying random phenomena
- Examples:
  - X = the outcome of a dice roll
  - X = the temperature tomorrow

Probability Mass Function

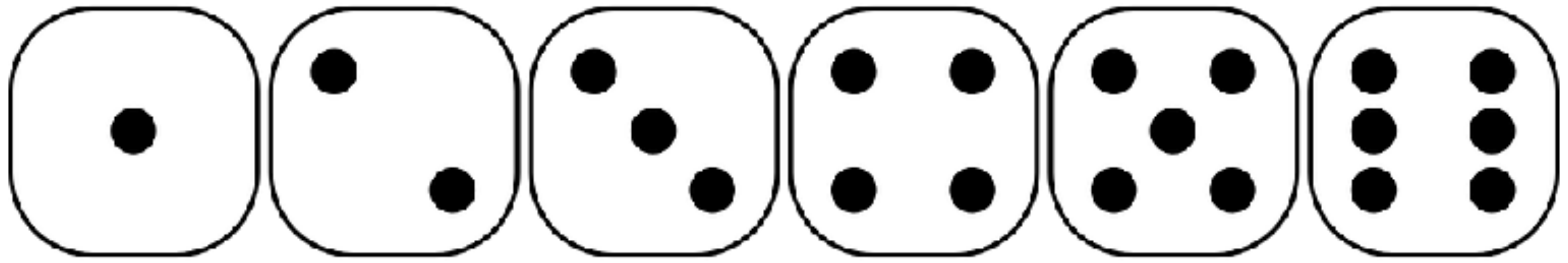
- Outcome space (or sample space) is {1, 2, 3, 4, 5, 6}
- p(x) = 1/6 for all x in {1, 2, 3, 4, 5, 6}
- Equivalently we write: p(x) = Pr(X = x) = P(X = x)
  (book uses Pr, also common to use P, we'll use both)



Random Variable: A variable that can take one of many possible values.



Sample space: The set of possible values for a random variable.



Notation

Probability!

$$P(X = x)$$

The name of the random variable

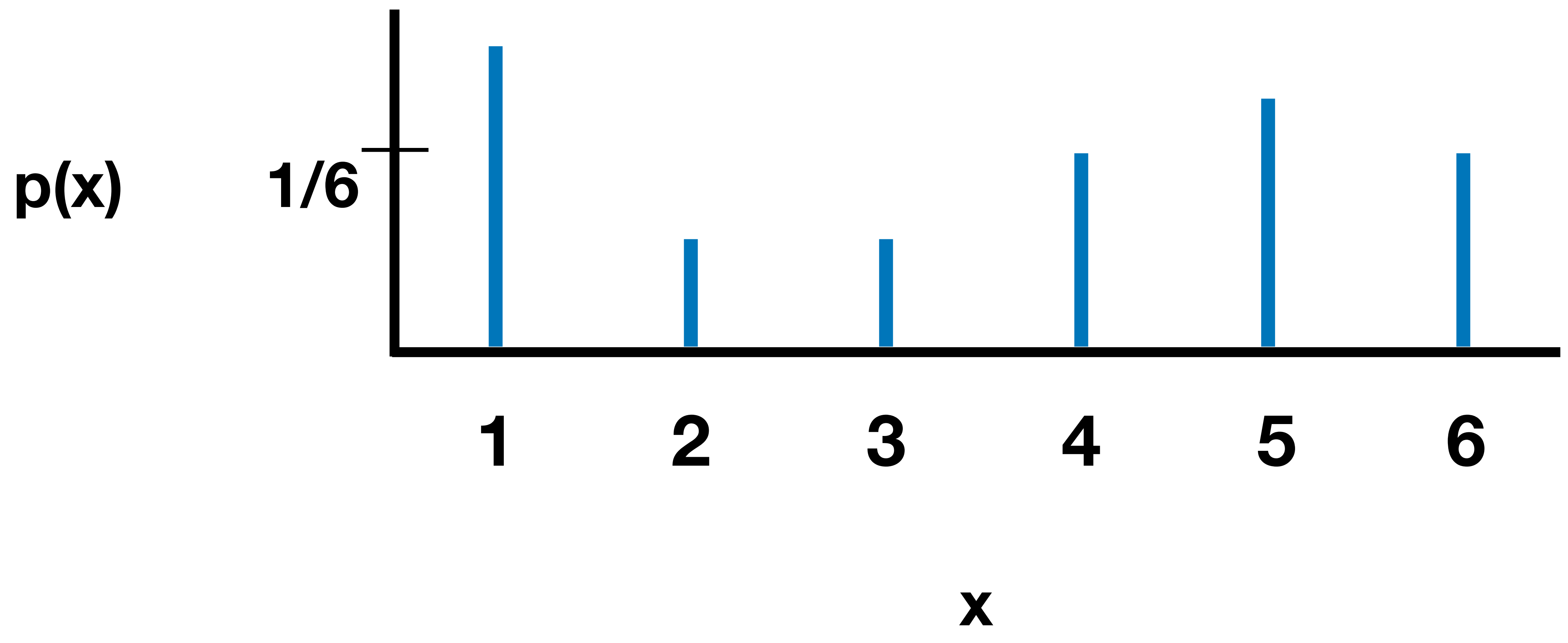
One of the elements in the sample space for X

"The probability that the random variable 'X' takes the value 'x'"

We use big P when we really want to be clear that we are talking about the probability of an RV X for a particular instance x
Little p is the PMF that defines those probabilities

Another PMF

- Outcome space is {1, 2, 3, 4, 5, 6}
- What does this PMF say?



Properties of the PMF

1. $p : \mathcal{X} \to [0,1]$

   i.e., $0 \leq p(x) \leq 1$

2. $$\sum_{x \in \mathcal{X}} p(x) = 1$$

e.g., $\mathcal{X} = \{1, 2, 3, 4, 5, 6\}$

Another example

- Imagine that you are a medical doctor, prescribing treatments to patients
- You give the treatment for 100 patients and note the outcome X = treatment outcome in {Bad, Neutral, Good}.
- 20 patients have a Bad outcome
- 50 a Neutral outcome
- 30 a Good outcome
- Obtain p(x) using 20/100, 50/100 and 30/100

$$p(x) = \begin{cases} 0.2 & \text{if } x \text{ is Bad} \\ 0.5 & \text{if } x \text{ is Neutral} \\ 0.3 & \text{if } x \text{ is Good.} \end{cases}$$

Another example

- Imagine that you are a medical doctor, prescribing treatments to patients
- You give the treatment for many patients and note the outcome X = treatment outcome in {Bad, Neutral, Good}. You find that you have the following probabilities:

$$p(x) = \begin{cases} 0.2 & \text{if } x \text{ is Bad} \\ 0.5 & \text{if } x \text{ is Neutral} \\ 0.3 & \text{if } x \text{ is Good.} \end{cases}$$

- But that's pretty stochastic…

How can we make this less stochastic?

- What if we knew something about the patient?
- What information could help?

$$p(x) = \begin{cases} 0.2 & \text{if } x \text{ is Bad} \\ 0.5 & \text{if } x \text{ is Neutral} \\ 0.3 & \text{if } x \text{ is Good.} \end{cases}$$

Conditional Probabilities

- X = outcome of treatment
- Y = age of the patient
- P(X = Good | Y = 22) is a different value than P(X = Good | Y = 78)

$$P(X = x \mid Y = y)$$

"The probability that the random variable 'X' takes the value 'x', given that the random variable 'Y' has taken the value 'y'"

The name of another random variable

One of the elements in the sample space for Y

Another example of conditional probabilities

- Often the value of a random variable is dependent on or correlated with another random variable

- Example: A store restocks on Wednesday, so the probability that they have pencils in stock depends on the day of the week

1	Wed	**Pencil Delivery!**
2	Thu	
3	Fri	
4	Sat	
5	Sun	
6	Mon	
7	Tues	

$P(\text{Pencils in stock} \mid \text{Tuesday}) = 0.2$ $\qquad$ $P(\text{Pencils in stock} \mid \text{Wednesday}) = 1.0$

"if" statement of probability

Same rules for conditional probabilities as for unconditioned probabilities

$$1.\ \ p(\cdot|Y=y):\mathcal{X}\to[0,1]$$

Could write it $p_{Y=y}(x)$

$$2.\ \ \sum_{x\in\mathcal{X}} p(x|Y=y)=1$$

e.g. If $P(\text{Pencils in stock} \mid \text{Tuesday}) = 0.2$, then
$P(\text{Pencils NOT in stock} \mid \text{Tuesday}) = 0.8$

Exercise 1

- Find $P(X = 3 \mid X = 3)$ and $P(X = 9 \mid X = 4)$.

Known

Known

Must be true

Impossible

Exercise 1: Answer

- Find $P(X = 3 \mid X = 3)$ and $P(X = 9 \mid X = 4)$.

Known

Must be true

$$P(X = 3 \mid X = 3) = 1$$

Multiple random variables

- We now have two RVs, X and Y and can reason about more than just p(x | y)
- We can ask about their joint probability: p(x, y)
- We might want to ask about the other conditional: p(y | x)

Back to the Pencils example

- Let X in {0, 1} (Pencils not in stock, Pencils in stock)
- Let Y in {Monday, Tuesday, …, Sunday} (Notice p(y) = 1/7 for all y)
- We can ask P(X = 1, Y = Tuesday) or P(Y = Tuesday | X = 1)
- Do you think P(Y = Wednesday | X = 1) > P(Y = Tuesday | X = 1)

1	Wed	Pencil Delivery!
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5	Sun	
6	Mon	
7	Tues	

$P(\text{Pencils in stock} \mid \text{Tuesday}) = 0.2$ $\quad$ $P(\text{Pencils in stock} \mid \text{Wednesday}) = 1.0$

"if" statement of probability

Chain Rule

- Can express joint probability using conditional probabilities
- $p(x, y) = p(x \mid y)\, p(y)$
- $p(x, y) = p(y \mid x)\, p(x)$

Back to the Pencils example

- Do you think P(Y = Wednesday, X = 1) > P(Y = Tuesday, X = 1) ?
- Why or why not? Use P(Y = Wednesday, X = 1) **=** P(Y = Wednesday | X = 1) P(X = 1)

1	Wed	**Pencil Delivery!**
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$P(\text{Pencils in stock} \mid \text{Tuesday}) = 0.2$ $\quad$ $P(\text{Pencils in stock} \mid \text{Wednesday}) = 1.0$

"if" statement of probability

Independence and Conditional Independence

- X and Y are **independent** if and only if $p(x, y) = p(x)\, p(y)$
  - equivalently, if $p(x \mid y) = p(x)$
  - recall: $p(x, y) = p(x \mid y)\, p(y)$
- X and Y are **conditionally independent**, given Z, if and only if $p(x, y \mid z) = p(x \mid z)\, p(y \mid z)$

Example of Independence

- Imagine you flip a (fair) coin twice
- Let X = Outcome of flip 1 in {H, T}
- Let Y = Outcome of flip 2 in {H, T}
- $p(x, y) = p(x)\, p(y) = 0.5 \times 0.5 = 0.25$
- i.e., $P(X = H, Y = H) = P(X = H)\, P(Y = H)$, and $P(X = T, Y = H) = P(X = T)\, P(Y = H)$, …

Example of Conditional Independence

- For the pencils example, imagine you had an RV Z where
  - Z = 0 if the day of the week is in the range Saturday to Tuesday
  - Z = 1 if the day of the week is in the range Wednesday to Friday
- Z is not independent of X (recall, X is whether Pencils are in stock or not)
  - P(Z = 0 | X = 1) not equal to P(Z = 0) (P(Z = 0) = 4/7 whereas P(Z = 0 | X = 1) < 4/7)
- Z is conditionally independent of X, given Y (Y is the day of the week)
  - $P(Z=0|X=1, Y=D) = P(Z=0|Y=D) = \begin{cases} 0 & \text{if D in Sat to Tues} \\ 1 & \text{if D in Wed to Fri} \end{cases}$

Marginalization

- If we have the joint distribution p(x, y), we can find the marginals p(x) and p(y)

$$p(x) = \sum_{y \in \mathcal{Y}} p(x, y) \qquad p(y) = \sum_{x \in \mathcal{X}} p(x, y)$$

Exercise 2 (Marginalization)

- Imagine someone gives you $P(X = 1 \mid Y = y)$ for y in {Monday,…, Sunday}
- You already know $p(y) = 1/7$
- How would you get $p(X = 1)$? (i.e, the probability that Pencils are in stock)

Exercise 3 (Conditional Probs)

- You flip a coin and get two heads in a row. What is the probability that your third coin flip also results in heads?

Exercise 4
(Formalizing probability questions)

- You roll two standard dice. One of the die shows a 3. What is the probability that the sum of the dice is greater than 7?