Probability, continued

CMPUT 267: Basics of Machine Learning

§2.2-2.4

Outline

1. Multiple Random Variables
2. Independence
3. Expectations and Moments

Multiple Variables

Suppose we observe both a die's number, and where it lands.

$\Omega = \{(left,1),(right,1),(left,2),(right,2),\ldots,(right,6)\}$

Example: $X =$ number with $\mathcal{X} = \{1,2,3,4,5,6\}$ and $Y =$ position, with $\mathcal{Y} = \{\text{left, right}\}$

May ask questions like $P(X = 1, Y = \text{left})$ or $P(X \geq 4, Y = \text{left})$

Joint Distribution

We typically model the **interactions** of different random variables.

Joint probability mass function: $p(x, y) = P(X = x, Y = y)$

$$\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) = 1$$

Example: $\mathcal{X} = \{0,1\}$ (young, old) and $\mathcal{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 50/100	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 10/100	P(X=1, Y=1) = 39/100

Questions About Multiple Variables

Example: $\mathscr{X} = \{0,1\}$ (young, old)   and $\mathscr{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 50/100	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 10/100	P(X=1, Y=1) = 39/100

- Are these two variables related at all? Or do they change **independently**?
- Given this distribution, can we determine the distribution over just Y?
  I.e., what is $P(Y = 1)$? (**marginal distribution**)
- If we knew something about one variable, does that tell us something about the distribution over the other? E.g., if I know $X = 0$ (person is young), does that tell me the **conditional probability** $P(Y = 1 \mid X = 0)$? (Prob. that person we know is young has arthritis)

Marginal Distribution for Y

$$p(Y=0) = \sum_{x \in \mathcal{X}} p(x,0) = \sum_{x \in \{\text{young,old}\}} p(x,0)$$

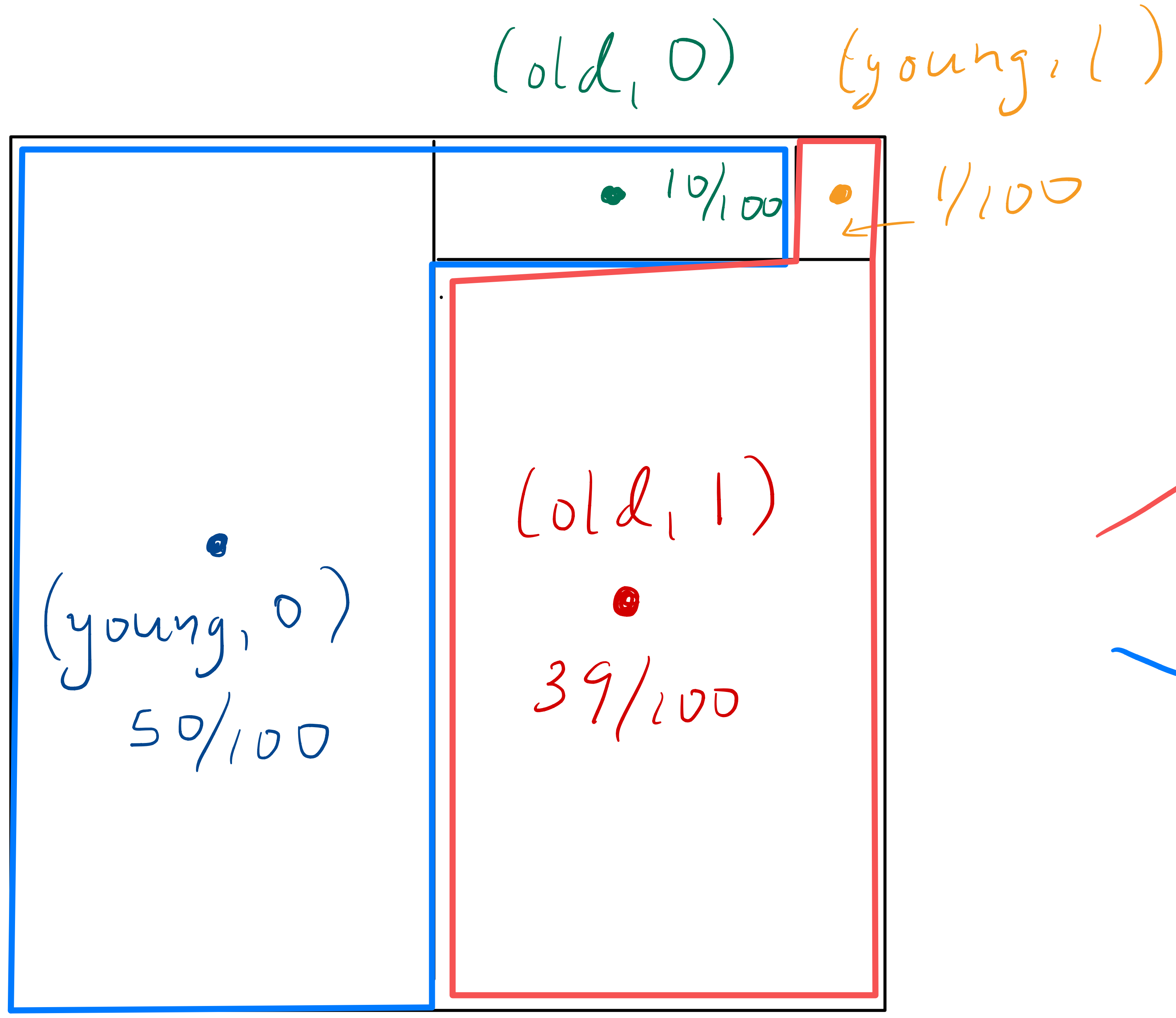
$$p(Y=1) = \sum_{x \in \mathcal{X}} p(x,1) = \sum_{x \in \{\text{young,old}\}} p(x,1)$$

Joint $p(x,y)$

Marginals = Area of subspace in joint event space

More generically

$$p(y) = \sum_{x \in \mathcal{X}} p(x,y)$$



$$p(Y=1) = \frac{39}{100} + \frac{1}{100} = 0.4$$

$$P(Y=0) = \frac{50}{100} + \frac{10}{100} = 0.6$$

Back to our example

Example: $\mathscr{X} = \{0,1\}$ (young, old) and $\mathscr{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 50/100	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 10/100	P(X=1, Y=1) = 39/100

- **Exercise**: Check if $\sum_{x \in \{0,1\}} \sum_{y \in \{0,1\}} p(x, y) = 1$
- **Exercise**: Compute marginal $p(x) = \sum_{y \in \{0,1\}} p(x, y)$

Back to our example (cont)

Example: $\mathscr{X} = \{0,1\}$ (young, old) and $\mathscr{Y} = \{0,1\}$ (no arthritis, arthritis)

	Y=0	**Y=1**
X=0	P(X=0,Y=0) = 50/100	P(X=0, Y=1) = 1/100
X=1	P(X=1, Y=0) = 10/100	P(X=1, Y=1) = 39/100

- **Exercise**: Check if $\sum_{x\in\{0,1\}} \sum_{y\in\{0,1\}} p(x,y) = 1/2 + 1/100 + 1/10 + 39/100 = 1$
- **Exercise**: Compute marginal $p(x=1) = \sum_{y\in\{0,1\}} p(x=1,y) = 49/100,$

$p(x=0) = 1 - p(x=1) = 51/100$

Can we get marginals for more than two variables?

PMFs and PDFs of Many Variables

In general, we can consider a d-dimensional random variable $X = (X_1, \ldots, X_d)$ with vector-valued outcomes $\mathbf{x} = (x_1, \ldots, x_d)$, with each x_i chosen from some $\mathcal{X}_i$. Then,

Discrete case:

$p : \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_d \to [0,1]$ is a **(joint) probability mass function** if

$$\sum_{x_1 \in \mathcal{X}_1} \sum_{x_2 \in \mathcal{X}_2} \cdots \sum_{x_d \in \mathcal{X}_d} p(x_1, x_2, \ldots, x_d) = 1$$

Continuous case:

$p : \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_d \to [0,\infty)$ is a **(joint) probability density function** if

$$\int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \cdots \int_{\mathcal{X}_d} p(x_1, x_2, \ldots, x_d)\, dx_1 dx_2 \ldots dx_d = 1$$

Rules of Probability Already Covered the Multidimensional Case

Outcome space is $\mathscr{X} = \mathscr{X}_1 \times \mathscr{X}_2 \times \ldots \times \mathscr{X}_d$

Outcomes are multidimensional variables $\mathbf{x} = [x_1, x_2, \ldots, x_d]$

Discrete case:

$p : \mathscr{X} \to [0,1]$ is a (joint) probability mass function if $\sum_{\mathbf{x} \in \mathscr{X}} p(\mathbf{x}) = 1$

Continuous case:

$p : \mathscr{X} \to [0,\infty)$ is a (joint) probability density function if $\int_{\mathscr{X}} p(\mathbf{x})\, d\mathbf{x} = 1$

But useful to recognize that we have multiple variables

Marginal Distributions

A **marginal distribution** is defined for a subset of X by summing or integrating out the remaining variables. (We will often say that we are "marginalizing over" or "marginalizing out" the remaining variables).

Discrete case:

$$p(x_i) = \sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_{i-1} \in \mathcal{X}_{i-1}} \sum_{x_{i+1} \in \mathcal{X}_{i+1}} \cdots \sum_{x_d \in \mathcal{X}_d} p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d)$$

e.g. $p(X_i = 1) = \sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_{i-1} \in \mathcal{X}_{i-1}} \sum_{x_{i+1} \in \mathcal{X}_{i+1}} \cdots \sum_{x_d \in \mathcal{X}_d} p(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d)$

Continuous:

$$p(x_i) = \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_{i-1}} \int_{\mathcal{X}_{i+1}} \cdots \int_{\mathcal{X}_d} p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d)\, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$$

Marginal Distributions

A **marginal distribution** is defined for a subset of X by summing or integrating out the remaining variables. (We will often say that we are "marginalizing over" or "marginalizing out" the remaining variables).

Discrete case: $p(x_i) = \sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_{i-1} \in \mathcal{X}_{i-1}} \sum_{x_{i+1} \in \mathcal{X}_{i+1}} \cdots \sum_{x_d \in \mathcal{X}_d} p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d)$

Continuous: $p(x_i) = \int_{\mathcal{X}_1} \cdots \int_{\mathcal{X}_{i-1}} \int_{\mathcal{X}_{i+1}} \cdots \int_{\mathcal{X}_d} p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d)\, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d$

Question: How do we get $p(x_i, x_j)$ for some i, j?

Question: Why p for $p(x_i)$ and $p(x_1, \ldots, x_d)$?

- They can't be the same function, they have different domains!

Are these really the same function?

- **No.** They're not the same function.
- But they are **derived** from the **same joint distribution**.
- So for brevity we will write

$$p(y \mid x) = \frac{p(x, y)}{p(x)}$$

- Even though it would be more precise to write something like

$$p_{Y|X}(y \mid x) = \frac{p(x, y)}{p_X(x)}$$

- We can tell which function we're talking about from context (i.e., arguments)

Conditional Distribution

Definition: **Conditional probability distribution**

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

This same equation will hold for the corresponding PDF or PMF:

$$p(y \mid x) = \frac{p(x, y)}{p(x)}$$

Question: if $p(x, y)$ is small, does that imply that $p(y \mid x)$ is small?

e.g., imagine x = arthritis and y = old

Chain Rule

From the definition of conditional probability:

$$\begin{aligned} p(y \mid x) &= \frac{p(x,y)}{p(x)} \\ \iff p(y \mid x)p(x) &= \frac{p(x,y)}{p(x)}p(x) \\ \iff p(y \mid x)p(x) &= p(x,y) \end{aligned}$$

This is called the **Chain Rule**.

Multiple Variable Chain Rule

The chain rule generalizes to multiple variables:

$$p(x, y, z) = p(x, y \mid z)p(z) = p(x \mid y, z)\underbrace{p(y \mid z)p(z)}_{p(y,z)}$$

Definition: **Chain rule**

$$\begin{aligned} p(x_1, \ldots, x_d) &= p(x_1 \mid x_2, \ldots, x_d)p(x_2 \mid x_3, \ldots, x_d)\ldots p(x_{d-1} \mid x_d)p(x_d) \\ &= p(x_d)\prod_{i=1}^{d-1} p(x_i \mid x_{i+1}, \ldots, x_d) \end{aligned}$$

The Order Does Not Matter

The RVs are not ordered, so we can write

$$
\begin{aligned}
p(x, y, z) &= p(x \mid y, z)p(y|z)p(z) \\
&= p(x \mid y, z)p(z|y)p(y) \\
&= p(y \mid x, z)p(x|z)p(z) \\
&= p(y \mid x, z)p(z|x)p(x) \\
&= p(z \mid x, y)p(y|x)p(x) \\
&= p(z \mid x, y)p(x|y)p(y)
\end{aligned}
$$

All of these probabilities are equal

Alternative Definition

Definition: Chain rule

$$p(x_1, \ldots, x_d) = p(x_1 \mid x_2, \ldots, x_d)p(x_2 \mid x_3, \ldots, x_d)\ldots p(x_{d-1} \mid x_d)p(x_d)$$
$$= p(x_d)\prod_{i=1}^{d-1} p(x_i \mid x_{i+1}, \ldots, x_d)$$

Definition: Chain rule

$$p(x_1, \ldots, x_d) = p(x_d \mid x_{d-1}, \ldots, x_1)p(x_{d-1} \mid x_{d-2}, \ldots, x_1)\ldots p(x_2 \mid x_1)p(x_1)$$
$$= p(x_1)\prod_{i=2}^{d} p(x_i \mid x_{i-1}, \ldots, x_1)$$

Bayes' Rule

From the chain rule, we have:

$$\begin{aligned} p(x, y) &= p(y \mid x)p(x) \\ &= p(x \mid y)p(y) \end{aligned}$$

- Often, $p(x \mid y)$ is easier to compute than $p(y \mid x)$
  - e.g., where x is **features** and y is **label**

Definition: Bayes' rule

$$p(y \mid x) = \frac{p(x \mid y)p(y)}{p(x)}$$

Bayes' Rule

- Bayes' rule is typically used to reason about our beliefs, given new information
- Example: a scientist might have a belief about the prevalence of cancer in smokers (Y), and update with new evidence (X)
- In ML: we have a belief over our estimator (Y), and we update with new data that is like new evidence (X)

Definition: Bayes' rule

$$p(y \mid x) = \frac{p(x \mid y)\, p(y)}{p(x)}$$

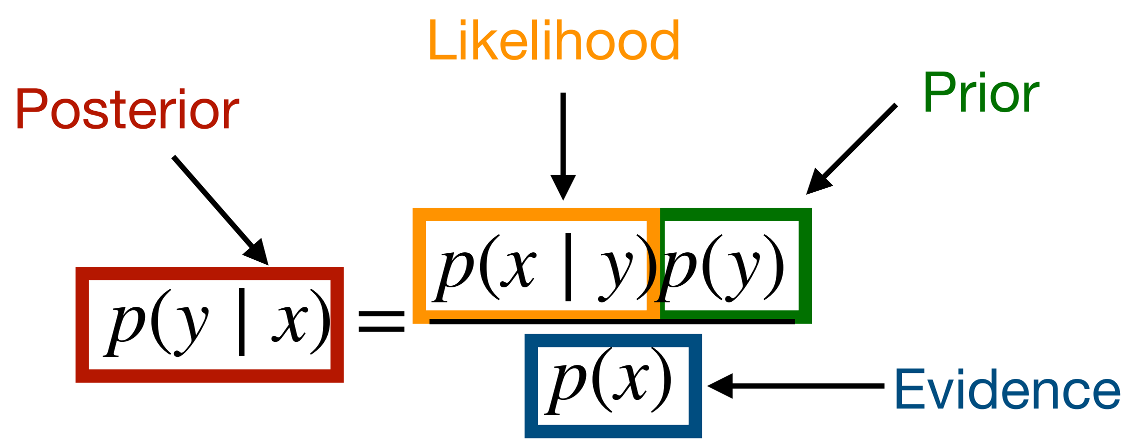
Posterior: $p(y \mid x)$

Likelihood: $p(x \mid y)$

Prior: $p(y)$

Evidence: $p(x)$

Example:
Disease Test



Example:

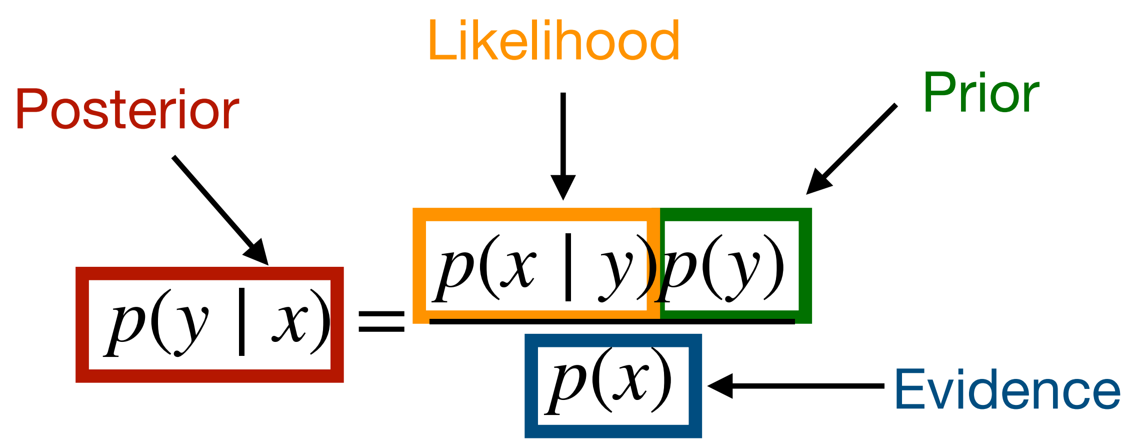
$$p(Test = pos \mid Dis = T) = 0.99$$
$$p(Test = pos \mid Dis = F) = 0.03$$
$$p(Dis = T) = 0.005$$

Mapping to the formula, let
X be Test
Y be presence of the Disease

Questions:

1. What is $p(Dis = F)$?
2. What is $p(Dis = T \mid Test = pos)$?

Example: Disease Test



Example:

$$p(Test = pos \mid Dis = T) = 0.99$$

$$p(Test = pos \mid Dis = F) = 0.03$$

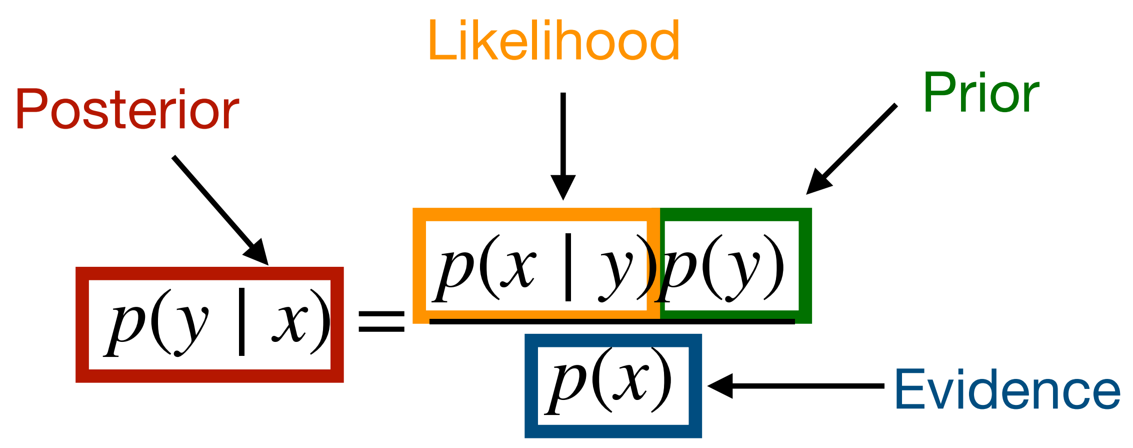
$$p(Dis = T) = 0.005$$

Questions:

1. What is $p(Dis = F)$?
2. What is $p(Dis = T \mid Test = pos)$?

$$p(Dis = F) = 1 - p(Dis = T) = 1 - 0.005 = 0.995$$

Example:
Disease Test



Example:

$$p(Test = pos \mid Dis = T) = 0.99$$
$$p(Test = pos \mid Dis = F) = 0.03$$
$$p(Dis = T) = 0.005$$

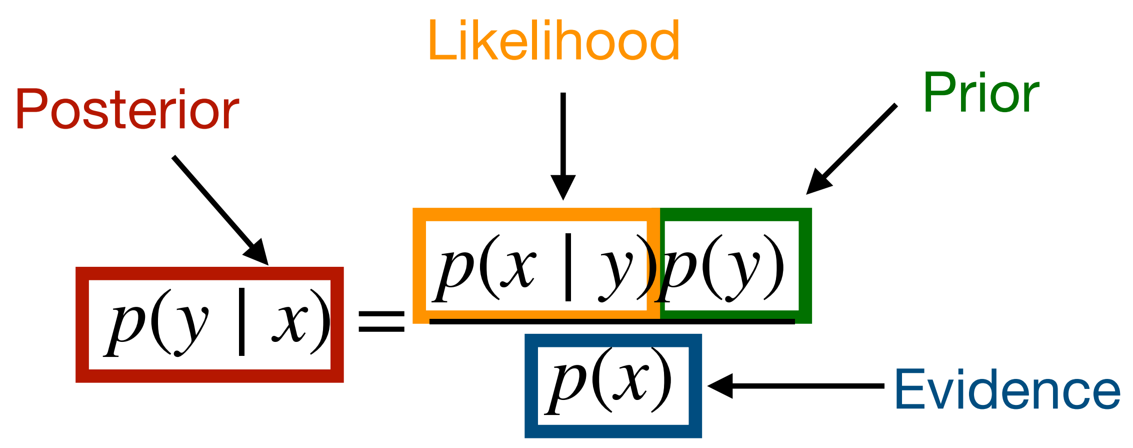
Questions:

1. What is $p(Dis = F)$?
2. What is $p(Dis = T \mid Test = pos)$?

$$p(Dis = T \mid Test = pos) = \frac{p(Test = pos \mid Dis = T)p(Dis = T)}{p(Test = pos)}$$

Need to compute this part

Example: Disease Test



Example:

$$p(Test = pos \mid Dis = T) = 0.99$$

$$p(Test = pos \mid Dis = F) = 0.03$$

$$p(Dis = T) = 0.005$$

Questions:

1. What is $p(Dis = F)$?
2. What is $p(Dis = T \mid Test = pos)$?

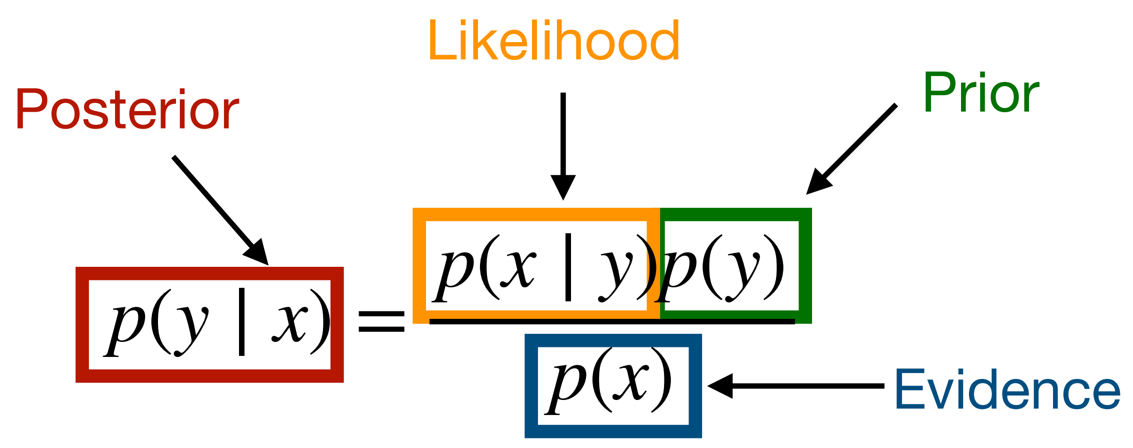
$$p(Test = pos) = \sum_{d \in \{T,F\}} p(Test = pos, d)$$

$$= p(Test = pos, D = F) + p(Test = pos, D = T)$$

$$= p(Test = pos \mid D = F)p(D = F) + p(Test = pos \mid D = T)p(D = T)$$

$$= 0.03 \times 0.995 + 0.99 \times 0.005 = 0.0348$$

Example:
Disease Test



Example:

$$p(Test = pos \mid Dis = T) = 0.99$$

$$p(Test = pos \mid Dis = F) = 0.03$$

$$p(Dis = T) = 0.005$$

Questions:

1. What is $p(Dis = F)$?
2. What is $p(Dis = T \mid Test = pos)$?

$$p(Test = pos) = 0.0348$$

$$p(Dis = T \mid Test = pos) = \frac{p(Test = pos \mid Dis = T)p(Dis = T)}{p(Test = pos)} = \frac{0.99 \times 0.005}{0.0348} \approx 0.142$$

Independence of Random Variables

Definition: X and Y are **independent** if:

$$p(x, y) = p(x)p(y)$$

X and Y are **conditionally independent given Z** if:

$$p(x, y \mid z) = p(x \mid z)p(y \mid z)$$

Example: Coins (Ex.7 in the course text)

- Suppose you have a biased coin: It does not come up heads with probability 0.5. Instead, it is more likely to come up heads.
- Let Z be the bias of the coin, with $\mathscr{Z} = \{0.3, 0.5, 0.8\}$ and probabilities $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$.
  - **Question:** What other outcome space could we consider?
  - **Question:** What kind of distribution is this?
  - **Question:** What other kinds of distribution could we consider?

Example: Coins (2)

- Now imagine I told you $Z = 0.3$ (i.e., probability of heads is 0.3)
- Let X and Y be two consecutive flips of the coin
- What is $P(X = Heads \mid Z = 0.3)$? What about $P(X = Tails \mid Z = 0.3)$?
- What is $P(Y = Heads \mid Z = 0.3)$? What about $P(Y = Tails \mid Z = 0.3)$?
- Is $P(X = x, Y = y \mid Z = 0.3) = P(X = x \mid Z = 0.3)P(Y = y \mid Z = 0.3)$?

Example: Coins (3)

- Now imagine we do not know Z
  - e.g., you randomly grabbed it from a bin of coins with probabilities $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$
- What is $P(X = Heads)$?

$$P(X = Heads) = \sum_{z \in \{0.3, 0.5, 0.8\}} P(X = Heads \,|\, Z = z) p(Z = z)$$

$$= P(X = Heads \,|\, Z = 0.3) p(Z = 0.3)$$

$$+ P(X = Heads \,|\, Z = 0.5) p(Z = 0.5)$$

$$+ P(X = Heads \,|\, Z = 0.8) p(Z = 0.8)$$

$$= 0.3 \times 0.7 + 0.5 \times 0.2 + 0.8 \times 0.1 = 0.39$$

Example: Coins (4)

- Now imagine we do not know Z
  - e.g., you randomly grabbed it from a bin of coins with probabilities $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$
- Is $P(X = Heads, Y = Heads) = P(X = Heads)p(Y = Heads)$?
  - For brevity, lets use h for Heads

$$
\begin{aligned}
P(X = h, Y = h) &= \sum_{z \in \{0.3, 0.5, 0.8\}} P(X = h, Y = h \mid Z = z)p(Z = z) \\
&= \sum_{z \in \{0.3, 0.5, 0.8\}} P(X = h \mid Z = z)P(Y = h \mid Z = z)p(Z = z)
\end{aligned}
$$

Example: Coins (4)

- $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$
- Is $P(X = Heads, Y = Heads) = P(X = Heads)p(Y = Heads)$?

$$P(X = h, Y = h) = \sum_{z \in \{0.3,0.5,0.8\}} P(X = h, Y = h \,|\, Z = z)p(Z = z)$$

$$= \sum_{z \in \{0.3,0.5,0.8\}} P(X = h \,|\, Z = z)P(Y = h \,|\, Z = z)p(Z = z)$$

$$= P(X = h \,|\, Z = 0.3)P(Y = h \,|\, Z = 0.3)p(Z = 0.3)$$

$$+P(X = h \,|\, Z = 0.5)P(Y = h \,|\, Z = 0.5)p(Z = 0.5)$$

$$+P(X = h \,|\, Z = 0.8)p(Y = h \,|\, Z = 0.8)p(Z = 0.8)$$

$$= 0.3 \times 0.3 \times 0.7 + 0.5 \times \times 0.5 \times 0.2 + 0.8 \times 0.8 \times 0.1$$

$$= 0.177 \neq 0.39 * 0.39 = 0.1521$$

Example: Coins (4)

- Let Z be the bias of the coin, with $\mathscr{Z} = \{0.3, 0.5, 0.8\}$ and probabilities $P(Z = 0.3) = 0.7$, $P(Z = 0.5) = 0.2$ and $P(Z = 0.8) = 0.1$.
- Let X and Y be two consecutive flips of the coin
- **Question:** Are X and Y conditionally independent given Z?
  - i.e., $P(X = x, Y = y \mid Z = z) = P(X = x \mid Z = z)P(Y = y \mid Z = z)$
- **Question:** Are X and Y independent?
  - i.e. $P(X = x, Y = y) = P(X = x)P(Y = y)$

The Distribution Changes Based on What We Know

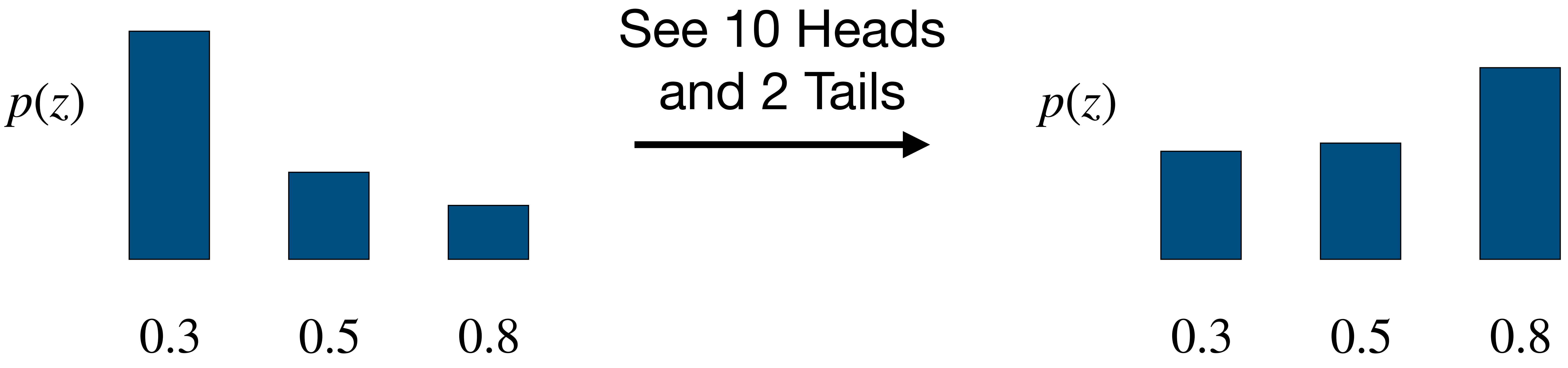
- The coin has some true bias z
- If we **know** that bias, we reason about $P(X = x \mid Z = z)$
  - Namely, the probability of x **given** we know the bias is z
- If we **do not know** that bias, then **from our perspective** the coin outcomes follows probabilities $P(X = x)$
  - The world still flips the coin with bias z
- Conditional independence is a property of the distribution we are reasoning about, not an objective truth about outcomes

A bit more intuition

- If we **do not know** that bias, then **from our perspective** the coin outcomes follows probabilities $P(X = x, Y = y)$
  - and X and Y are correlated
- If we know $X = h$, do we think it's more likely $Y = h$? i.e., is $P(X = h, Y = h) > P(X = h, Y = t)$?

My brain hurts, why do I need to know about coins?

- i.e., how is this relevant
- Let's imagine you want to infer (or learn) the bias of the coin, from data
  - data in this case corresponds to a sequence of flips $X_1, X_2, \ldots, X_n$
- You can ask: $P(Z = z \mid X_1 = H, X_2 = H, X_3 = T, \ldots, X_n = H)$



More uses for independence and conditional independence

- If I told you X = roof type was **independent** of Y = house price, would you use X as a feature to predict Y?
- Imagine you want to predict Y = Has Lung Cancer and you have an indirect correlation with X = Location since in Location 1 more people smoke on average. If you could measure Z = Smokes, then X and Y would be **conditionally independent** given Z.
    - Suggests you could look for such causal variables, that explain these correlations
- We will see the utility of conditional independence for learning models

Expected Value

The expected value of a random variable is the **weighted average** of that variable over its domain.

Definition: Expected value of a random variable

$$\mathbb{E}[X] = \begin{cases} \sum_{x \in \mathcal{X}} x p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} x p(x)\, dx & \text{if } X \text{ is continuous.} \end{cases}$$

Relationship to Population Average and Sample Average

- Or Population Mean and Sample Mean
- Population Mean = Expected Value, Sample Mean estimates this number
  - e.g., Population Mean = average height of the entire population
- For RV X = height, p(x) gives the probability that a randomly selected person has height x
- Sample average: you randomly sample n heights from the population
  - implicitly you are sampling heights proportionally to p
- As n gets bigger, the sample average approaches the true expected value

Connection to Sample Average

- Imagine we have a biased coin, p(x = 1) = 0.75, p(x = 0) = 0.25
- Imagine we flip this coin 1000 times, and see (x = 1) 700 times
- The sample average is

$$\frac{1}{1000}\sum_{i=1}^{1000} x_i = \frac{1}{1000}\left[\sum_{i:x_i=0} x_i + \sum_{i:x_i=1} x_i\right] = 0 \times \frac{300}{1000} + 1 \times \frac{700}{1000} == 0 \times 0.3 + 1 \times 0.7 = 0.7$$

- The true expected value is

$$\sum_{x\in\{0,1\}} p(x)x = 0 \times p(x=0) + 1p(x=1) = 0 \times 0.25 + 1 \times 0.75 = 0.75$$

Expected Value with Functions

The expected value of a function $f : \mathcal{X} \to \mathbb{R}$ of a random variable is the **weighted average** of that function's value over the domain of the variable.

Definition: Expected value of a function of a random variable

$$\mathbb{E}[f(X)] = \begin{cases} \sum_{x \in \mathcal{X}} f(x)p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} f(x)p(x)\, dx & \text{if } X \text{ is continuous.} \end{cases}$$

Example:

Suppose you get \$10 if heads is flipped, or lose \$3 if tails is flipped.
What are your winnings **on expectation**?

Expected Value Example

Example:
Suppose you get \$10 if heads is flipped, or lose \$3 if tails is flipped.
What are your winnings **on expectation**?

X is the outcome of the coin flip, 1 for heads and 0 for tails

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \\ 10 & \text{if } x = 1 \end{cases}$$

$Y = f(X)$ is a new random variable

$$\mathbb{E}[Y] = \mathbb{E}[f(X)] = \sum_{x \in \mathcal{X}} f(x)p(x) = f(0)p(0) + f(1)p(1) = .5 \times 3 + .5 \times 10 = 6.5$$

One More Example

Suppose X is the outcome of a dice role

$$f(x) = \begin{cases} -1 & \text{if } x \leq 3 \\ 1 & \text{if } x \geq 4 \end{cases}$$

$Y = f(X)$ is a new random variable. We see $Y = -1$ each time we observe 1, 2 or 3.

We see $Y = 1$ each time we observe 4, 5, or 6.

$$\mathbb{E}[Y] = \mathbb{E}[f(X)] = \sum_{x \in \mathcal{X}} f(x)p(x)$$

$$= (-1)\Big(p(X = 1) + p(X = 2) + p(X = 3)\Big)$$

$$+ \ (1)\Big(p(X = 4) + p(X = 5) + p(X = 6)\Big)$$

One More Example

Suppose X is the outcome of a dice role

$$f(x) = \begin{cases} -1 & \text{if } x \leq 3 \\ 1 & \text{if } x \geq 4 \end{cases}$$

$Y = f(X)$ is a new random variable. We see $Y = -1$ each time we observe 1, 2 or 3.
We see $Y = 1$ each time we observe 4, 5, or 6.

$$\mathbb{E}[Y] = \mathbb{E}[f(X)] = \sum_{x \in \mathcal{X}} f(x)p(x) = \sum_{y \in \{-1,1\}} yp(y)$$

$$p(Y = -1) = p(X = 1) + p(X = 2) + p(X = 3) = 0.5$$

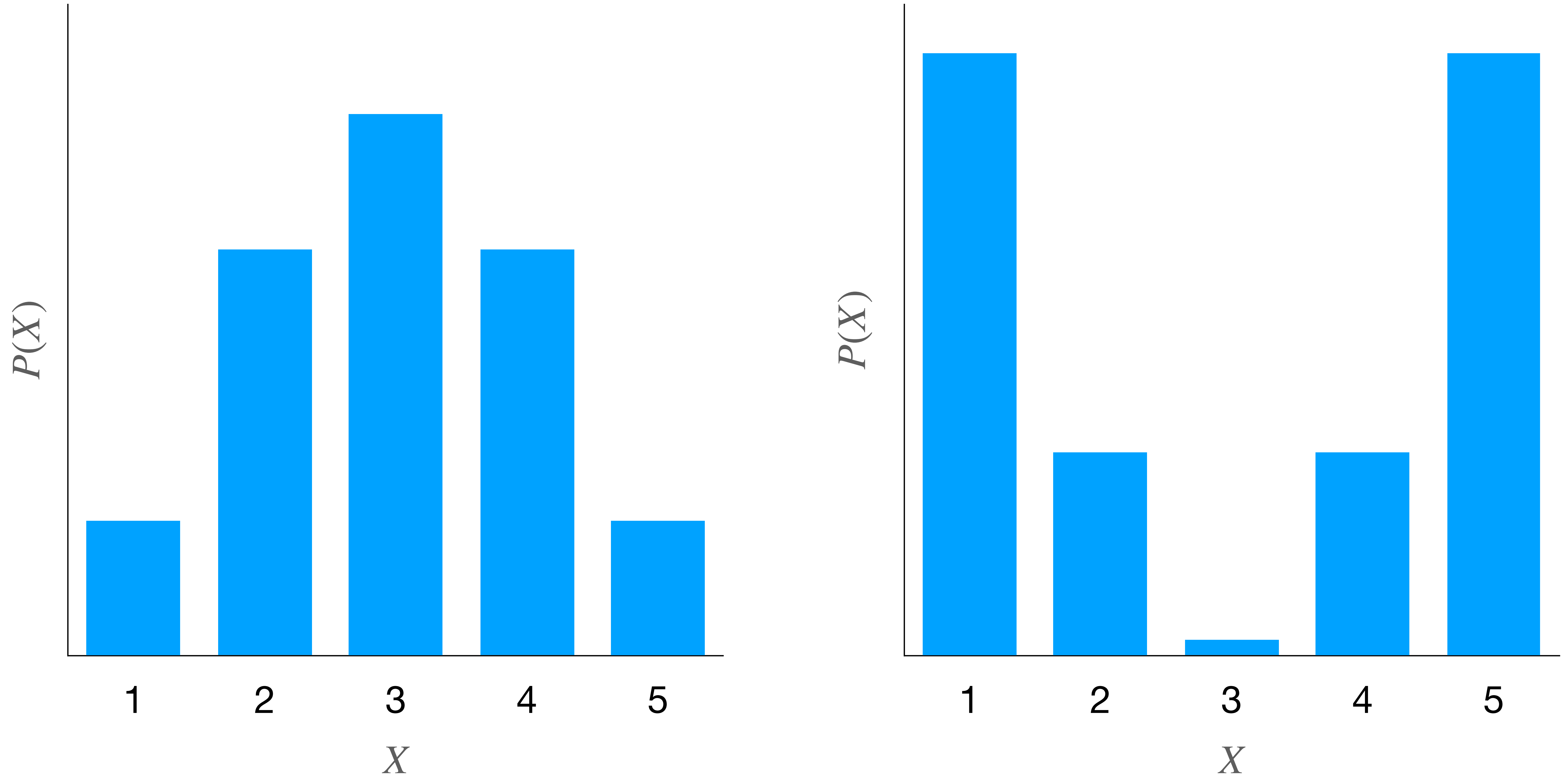
$$p(Y = 1) = p(X = 4) + p(X = 5) + p(X = 6) = 0.5$$

$$= (-1)\Big(p(X = 1) + p(X = 2) + p(X = 3)\Big)$$

$$+ \ (1)\Big(p(X = 4) + p(X = 5) + p(X = 6)\Big) \qquad = -1(0.5) + 1(0.5)$$

Summing over x with p(x) is equivalent, and simpler (no need to infer p(y))

Expected Value is a Lossy Summary



$$\mathbb{E}[X] = 3$$
$$\mathbb{E}[X^2] \simeq 10$$

$$\mathbb{E}[X] = 3$$
$$\mathbb{E}[X^2] \simeq 12$$

Conditional Expectations

Definition:

The **expected value of Y conditional on $X = x$** is

$$\mathbb{E}[Y \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} y p(y \mid x) & \text{if } Y \text{ is discrete,} \\ \int_{\mathcal{Y}} y p(y \mid x)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$$

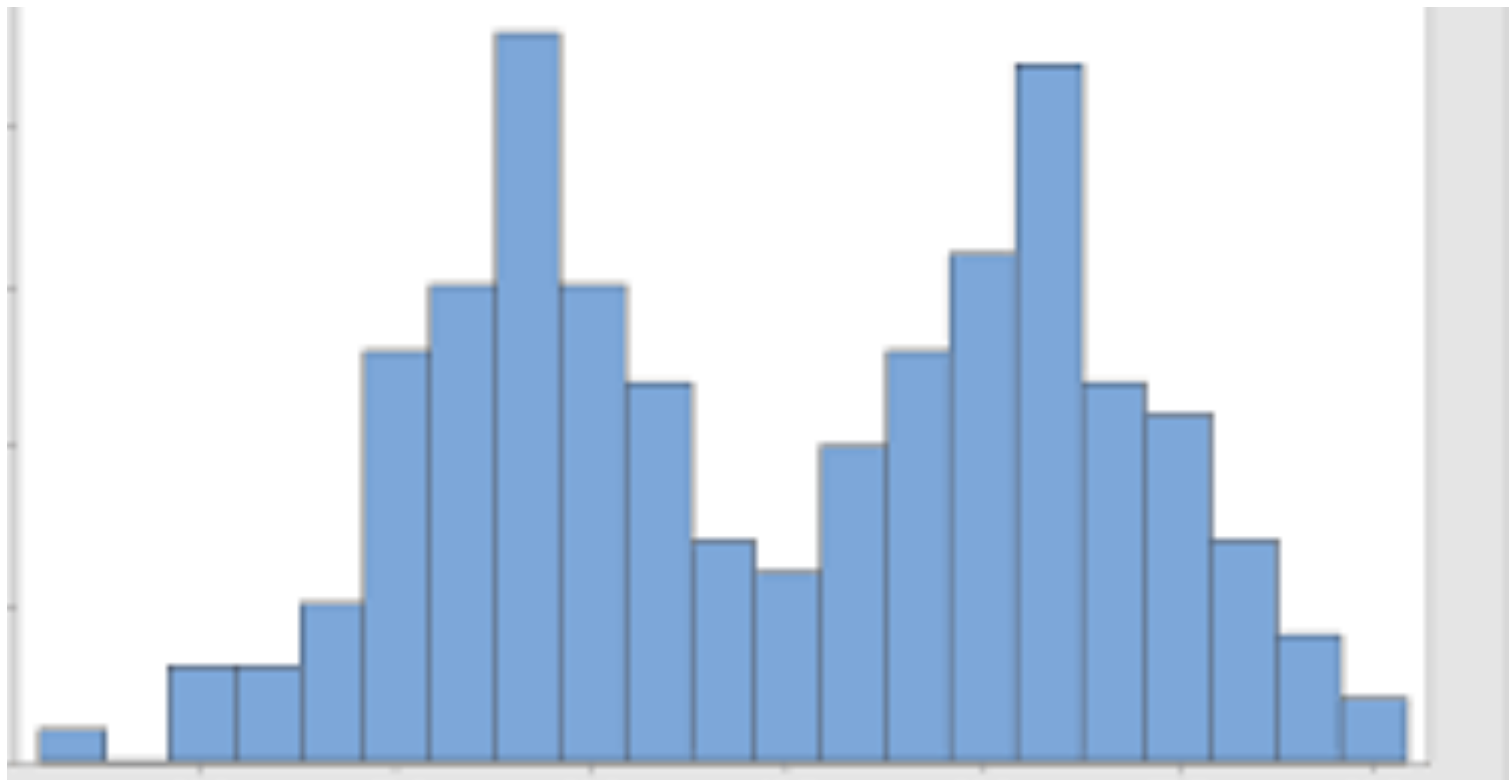
Conditional Expectation Example

- X is the type of a book, 0 for fiction and 1 for non-fiction
  - $p(X = 1)$ is the proportion of all books that are non-fiction
- Y is the number of pages
  - $p(Y = 100)$ is the proportion of all books with 100 pages
- $\mathbb{E}[Y|X = 0]$ is different from $\mathbb{E}[Y|X = 1]$
  - e.g. $\mathbb{E}[Y|X = 0] = 70$ is different from $\mathbb{E}[Y|X = 1] = 150$
- Another example: $\mathbb{E}[X|Z = 0.3]$ the expected outcome of the coin flip given that the bias is 0.3 ($\mathbb{E}[X|Z = 0.3] = 0 \times 0.7 + 1 \times 0.3 = 0.3$)

Conditional Expectation Example (cont)

- What do we mean by $p(y \mid X = 0)$? How might it differ from $p(y \mid X = 1)$

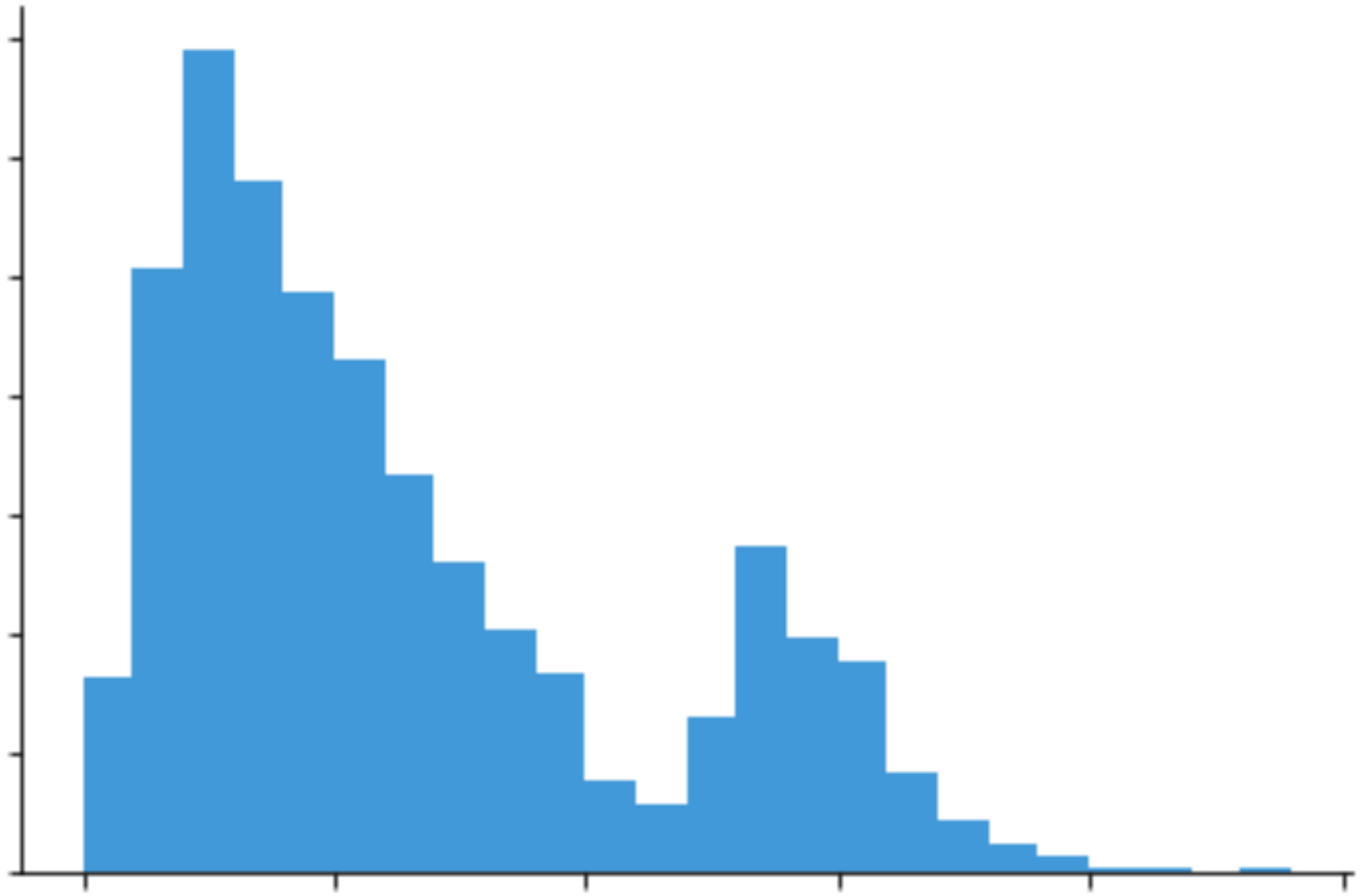
p(y) for X = 0, fiction books



Lots of shorter books

Lots of medium length books

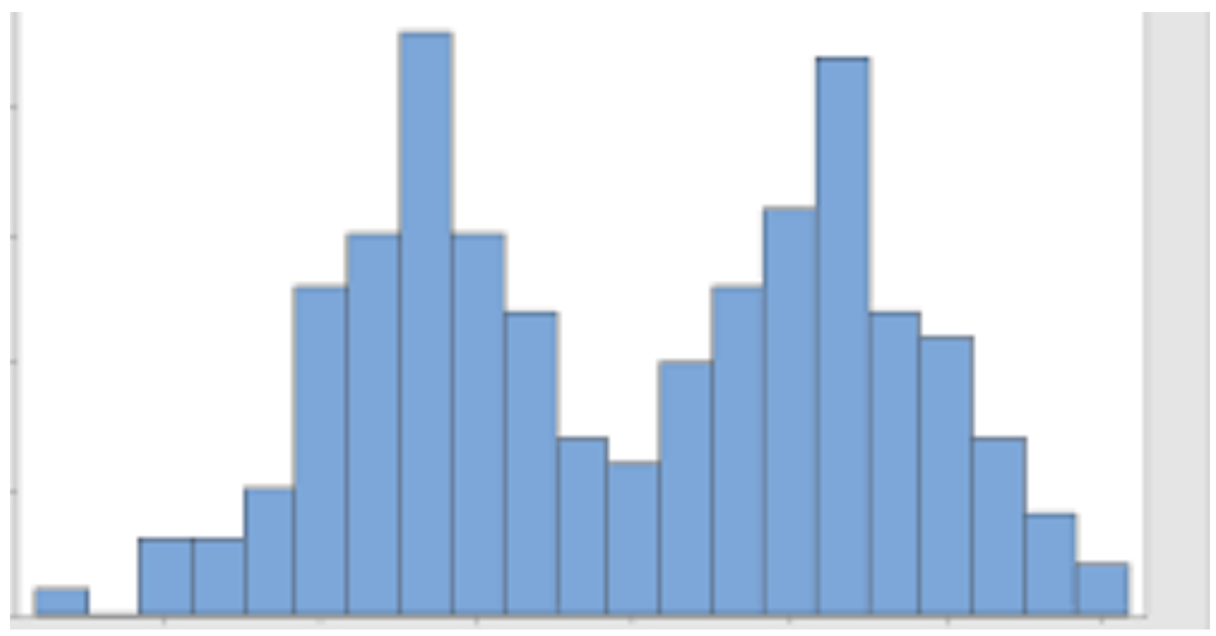
p(y) for X = 1, nonfiction books

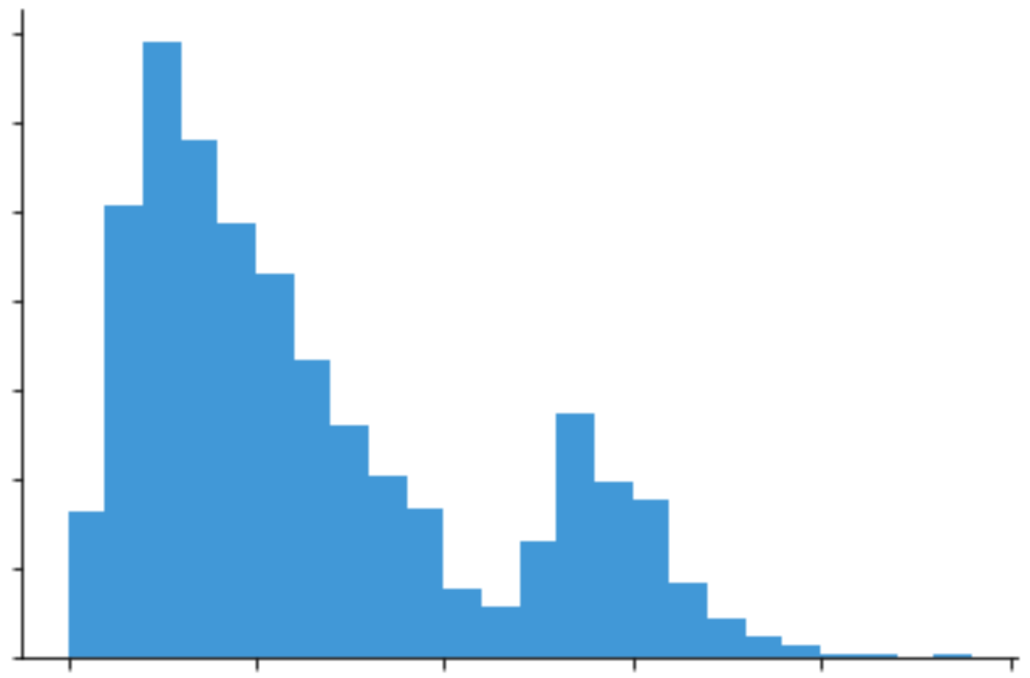


A long tail, a few very long books

Conditional Expectation Example (cont)

- What do we mean by $p(y \mid X = 0)$? How might it differ from $p(y \mid X = 1)$





- $\mathbb{E}[Y \mid X = 0]$ is the expectation over Y under distribution $p(y \mid X = 0)$
- $\mathbb{E}[Y \mid X = 1]$ is the expectation over Y under distribution $p(y \mid X = 1)$

Another way to Write Conditional Expectations

The **expected value of Y conditional on $X = x$** is

$$\mathbb{E}[Y \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} y p(y \mid x) & \text{if } Y \text{ is discrete,} \\ \int_{\mathcal{Y}} y p(y \mid x)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$$

Let $p_x(y) \doteq p(y \mid x)$, $\mathbb{E}[Y \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} y p_x(y) & \text{if } Y \text{ is discrete,} \\ \int_{\mathcal{Y}} y p_x(y)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$

Conditional Expectations

Definition:
The **expected value of Y conditional on $X = x$** is

$$\mathbb{E}[Y \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} y p(y \mid x) & \text{if } Y \text{ is discrete,} \\ \int_{\mathcal{Y}} y p(y \mid x)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$$

Question: What is $\mathbb{E}[Y \mid X]$?

Conditional Expectations

Definition:

The **expected value of Y conditional on $X = x$** is

$$\mathbb{E}[Y \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} y p(y \mid x) & \text{if } Y \text{ is discrete,} \\ \int_{\mathcal{Y}} y p(y \mid x)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$$

Question: What is $\mathbb{E}[Y \mid X]$?

Answer: $Z = \mathbb{E}[Y \mid X]$ is a random variable, $z = \mathbb{E}[Y \mid X = x]$ is an outcome

Properties of Expectations

- Linearity of expectation:
  - $\mathbb{E}[cX] = c\mathbb{E}[X]$ for all constant c
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Products of expectations of **independent** random variables X, Y:
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Law of Total Expectation:
  - $\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] = \mathbb{E}[Y]$
- **Question:** How would you prove these?

Linearity of Expectation

$$\mathbb{E}[X+Y] = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)(x+y)$$
$$= \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)(x+y)$$
$$= \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)x + \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)y$$

$$\sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)x = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)x$$
$$= \sum_{x\in\mathcal{X}} x \sum_{y\in\mathcal{Y}} p(x,y) \quad \triangleright\, p(x) = \sum_{y\in\mathcal{Y}} p(x,y)$$
$$= \sum_{x\in\mathcal{X}} xp(x)$$
$$= \mathbb{E}[X]$$

Linearity of Expectation

$$\begin{aligned}\mathbb{E}[X+Y] &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)(x+y)\\ &= \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)(x+y)\\ &= \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)x + \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)y\\ &= \mathbb{E}[X]+\mathbb{E}[Y]\end{aligned}$$

$$\begin{aligned}\sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)x &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)x\\ &= \sum_{x\in\mathcal{X}} x \sum_{y\in\mathcal{Y}} p(x,y) \qquad \triangleright\ p(x) = \sum_{y\in\mathcal{Y}} p(x,y)\\ &= \sum_{x\in\mathcal{X}} xp(x)\\ &= \mathbb{E}[X]\end{aligned}$$

What if the RVs are continuous?

$$
\begin{aligned}
\mathbb{E}[X+Y] &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)(x+y) \\
&= \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)(x+y) \\
&= \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)x + \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p(x,y)y \\
&= \mathbb{E}[X] + \mathbb{E}[Y]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X+Y] &= \int_{\mathcal{X}\times\mathcal{Y}} p(x,y)(x+y)d(x,y) \\
&= \int_{\mathcal{Y}}\int_{\mathcal{X}} p(x,y)(x+y)dxdy \\
&= \int_{\mathcal{Y}}\int_{\mathcal{X}} p(x,y)xdxdy + \int_{\mathcal{Y}}\int_{\mathcal{X}} p(x,y)ydxdy \\
&= \int_{\mathcal{X}} x\int_{\mathcal{Y}} p(x,y)dydx + \int_{\mathcal{Y}} y\int_{\mathcal{X}} p(x,y)dxdy \\
&= \int_{\mathcal{X}} xp(x)dx + \int_{\mathcal{Y}} yp(y)dy \\
&= \mathbb{E}[X] + \mathbb{E}[Y]
\end{aligned}
$$

Properties of Expectations

- Linearity of expectation:
  - $\mathbb{E}[cX] = c\mathbb{E}[X]$ for all constant c
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Products of expectations of **independent** random variables X, Y:
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Law of Total Expectation:
  - $\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] = \mathbb{E}[Y]$
- Notice: $f(x) = E[Y|X = x]$
  $\mathbb{E}[f(X)] = \mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] = \mathbb{E}[Y]$

$$\mathbb{E}[Y] = \sum_{y \in \mathcal{Y}} yp(y)$$ def. E[Y]

$$= \sum_{y \in \mathcal{Y}} y \sum_{x \in \mathcal{X}} p(x, y)$$ def. marginal distribution

$$= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} yp(x, y)$$ rearrange sums

$$= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} yp(y \mid x)p(x)$$ Chain rule

$$= \sum_{x \in \mathcal{X}} \left(\sum_{y \in \mathcal{Y}} yp(y \mid x) \right) p(x)$$

$$= \sum_{x \in \mathcal{X}} \left(\mathbb{E}[Y \mid X = x] \right) p(x)$$ def. E[Y | X = x]

$$= \color{red}{\sum_{x \in \mathcal{X}}} \left(\mathbb{E}[Y \mid X = x] \right) \color{red}{p(x)}$$

$$= \mathbb{E}\left(\mathbb{E}[Y \mid X] \right) \blacksquare$$ def. expected value of function

Variance

Definition: The **variance** of a random variable is

$$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right].$$

Equivalently,

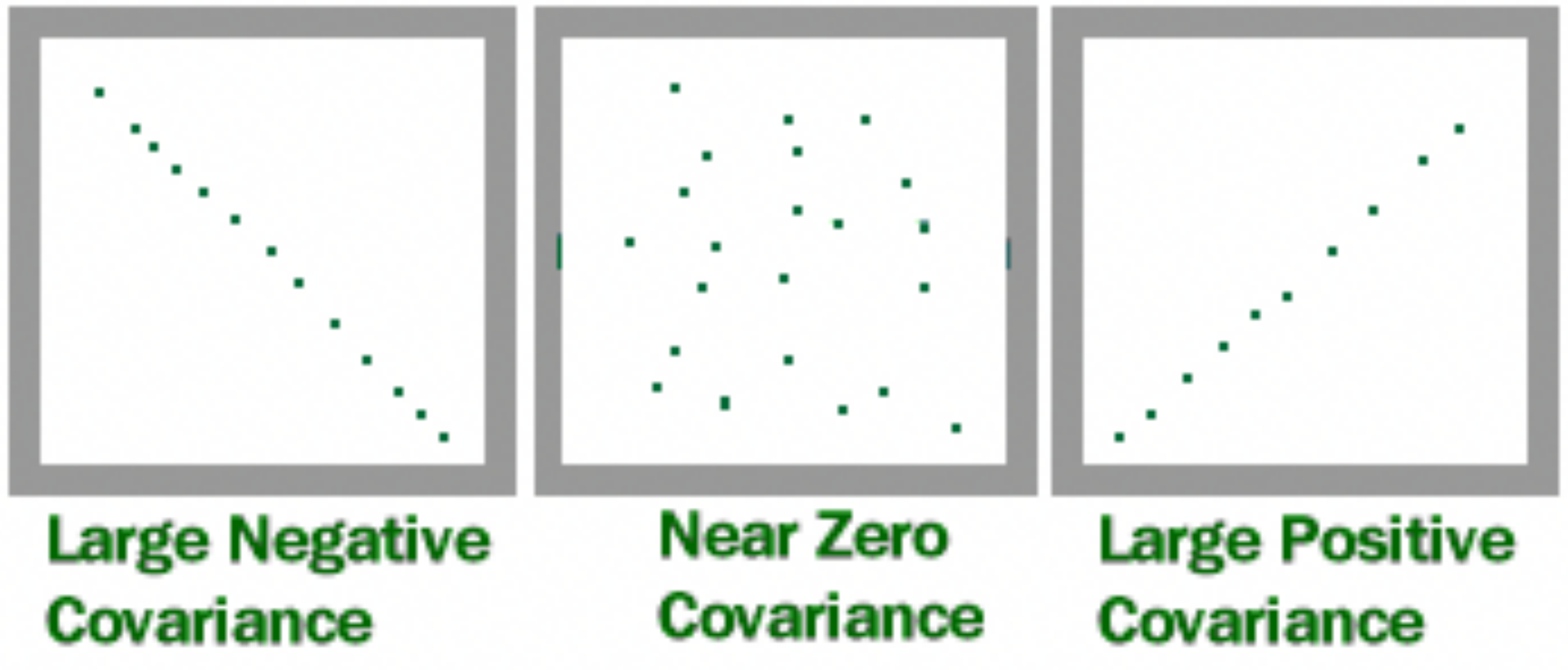
$$\text{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$$

(**Exercise:** Show that this is true)

Covariance

Definition: The covariance of two random variables is

$$\text{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\,.$$

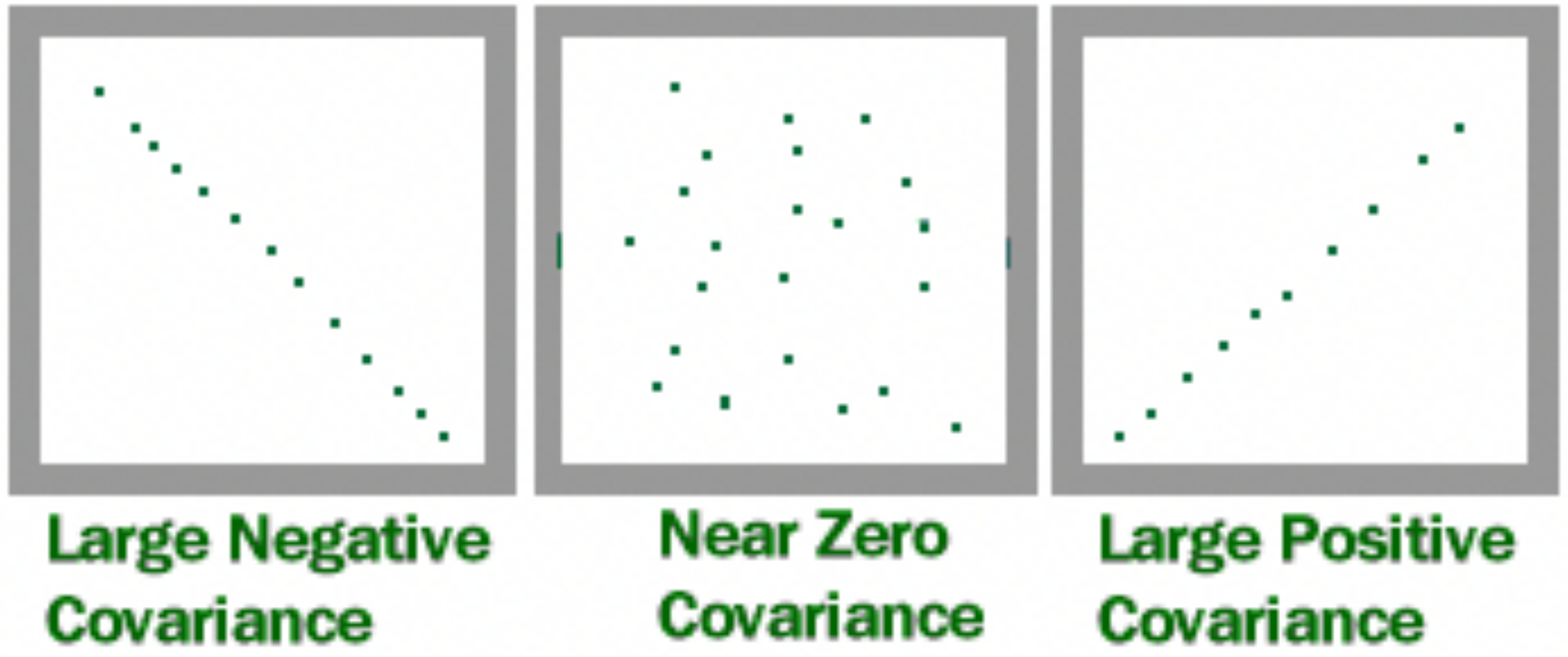


Question: What is the range of $\text{Cov}(X, Y)$?

Correlation

Definition: The **correlation** of two random variables is

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$



Question: What is the range of $\text{Corr}(X, Y)$?

hint: $\text{Var}(X) = \text{Cov}(X, X)$

Properties of Variances

- Var$[c] = 0$ for constant c
- Var$[cX] = c^2$Var$[X]$ for constant c
- Var$[X + Y] =$ Var$[X] +$ Var$[Y] + 2$Cov$[X, Y]$
- For **independent** X, Y,
  Var$[X + Y] =$ Var$[X] +$ Var$[Y]$ (**why?**)

Independence and Decorrelation

- Recall if X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Independent RVs have zero correlation (**why?**)

    hint: Cov$[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

- Uncorrelated RVs (i.e., Cov$(X, Y) = 0$) **might be dependent** (i.e., $p(x, y) \neq p(x)p(y)$).
    - Correlation (**Pearson's correlation coefficient**) shows linear relationships; but can miss nonlinear relationships
    - **Example:** $X \sim$ Uniform$\{-2, -1, 0, 1, 2\}$, $Y = X^2$
        - $\mathbb{E}[XY] = .2(-2 \times 4) + .2(2 \times 4) + .2(-1 \times 1) + .2(1 \times 1) + .2(0 \times 0)$
        - $\mathbb{E}[X] = 0$
        - So $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0 - 0\mathbb{E}[Y] = 0$

Summary

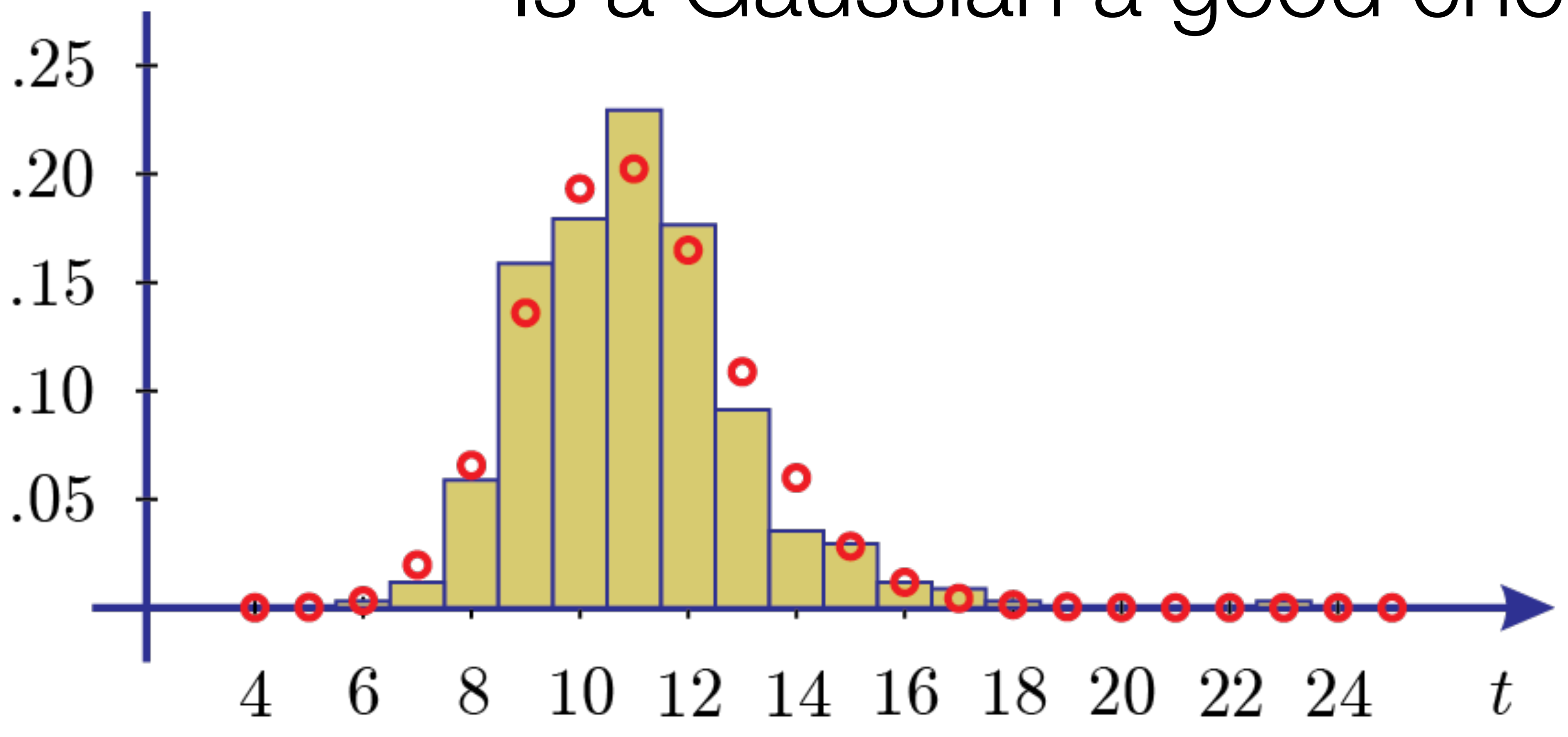
- **Random variables** takes different values with some probability
- The value of one variable can be informative about the value of another
  - Distributions of multiple random variables are described by the **joint** probability distribution (joint PMF or joint PDF)
  - You can have a new distribution over one variable when you **condition** on the other
- The **expected value** of a random variable is an **average** over its values, **weighted** by the probability of each value
- The **variance** of a random variable is the expected squared distance from the mean
- The **covariance** and **correlation** of two random variables can summarize how changes in one are informative about changes in the other.

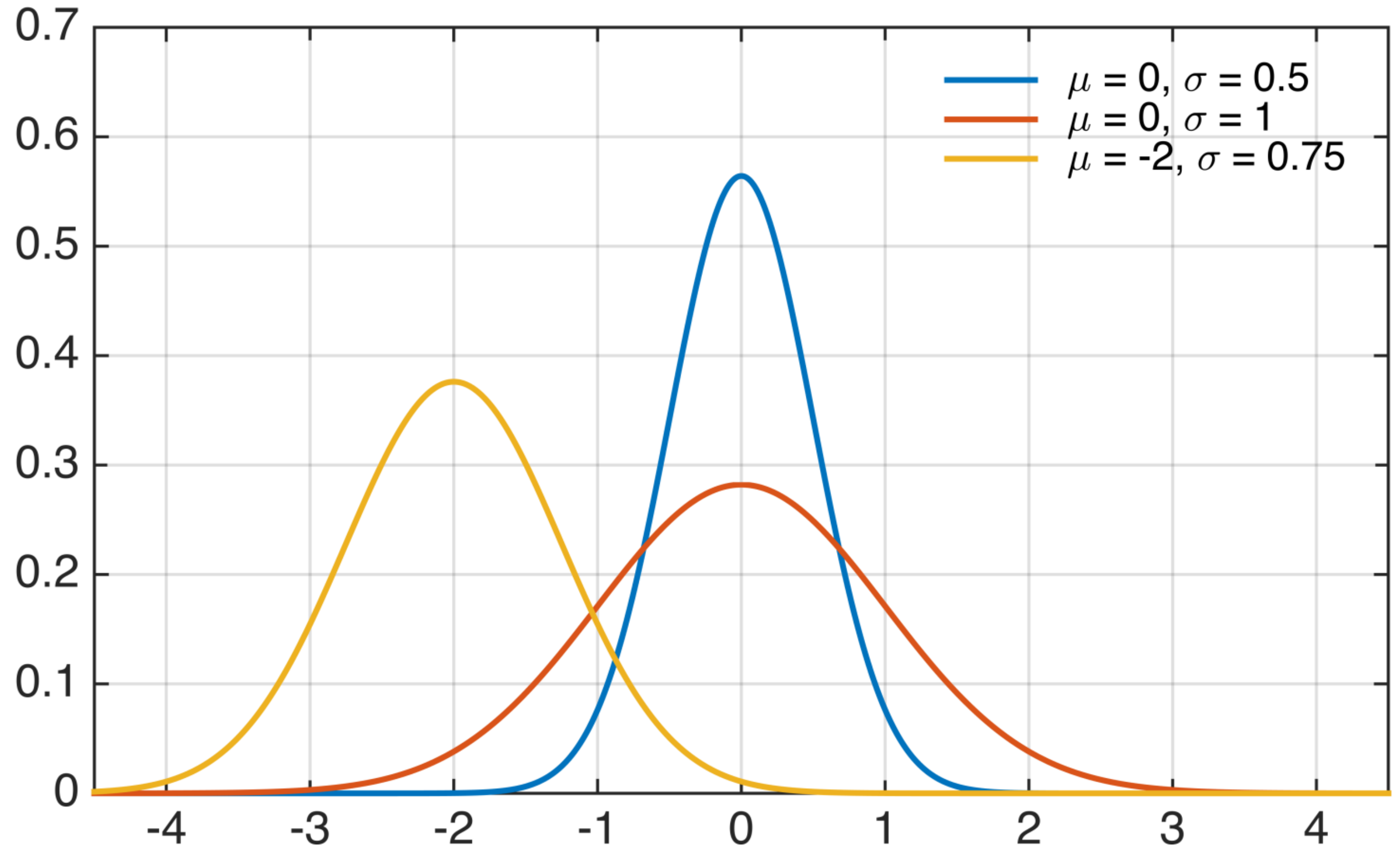
Exercise applying your knowledge

- Let's revisit the commuting example, and assume we collect continuous commute times
- We want to model commute time as a Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

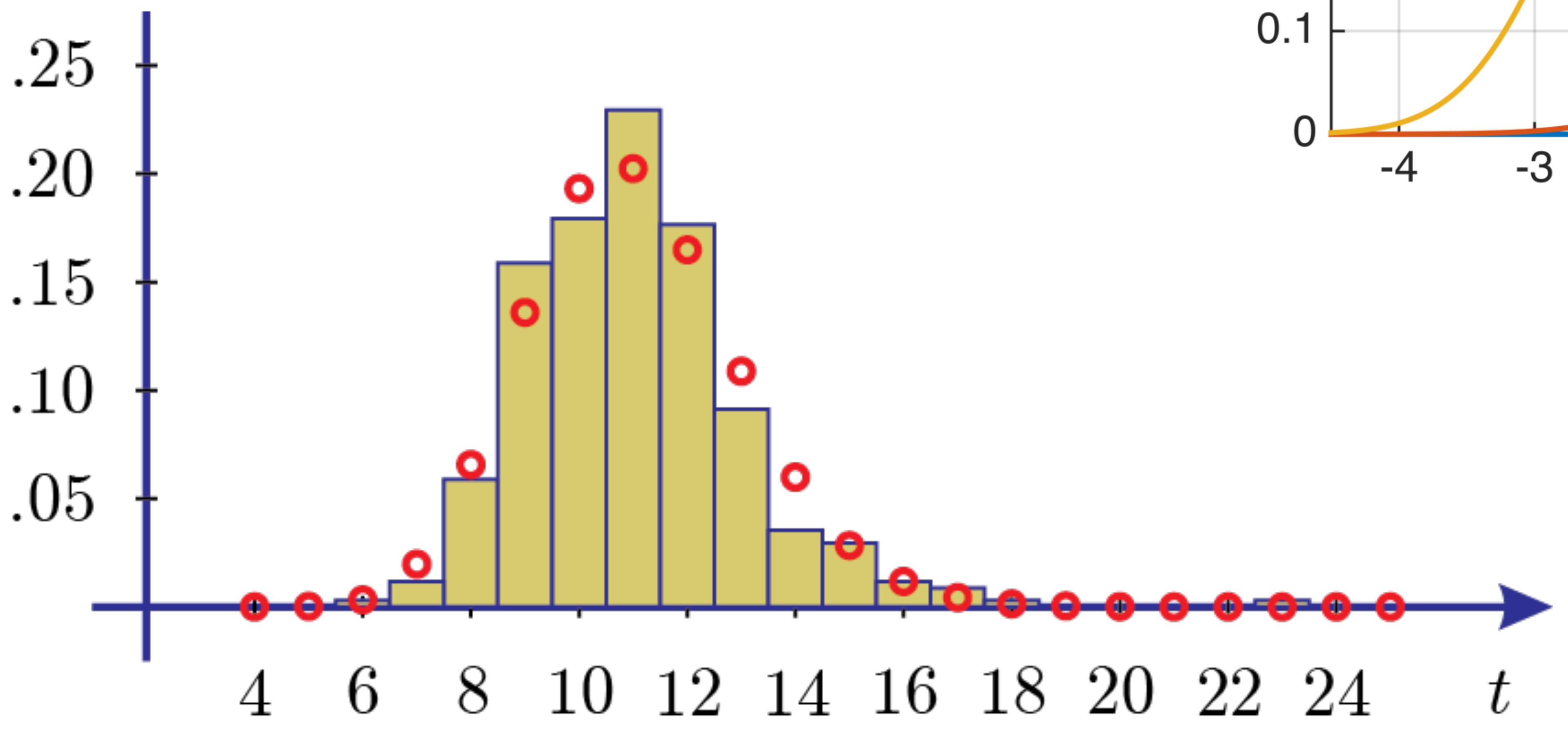
- What parameters do I have to specify (or learn) to model commute times with a Gaussian?
- Is a Gaussian a good choice?

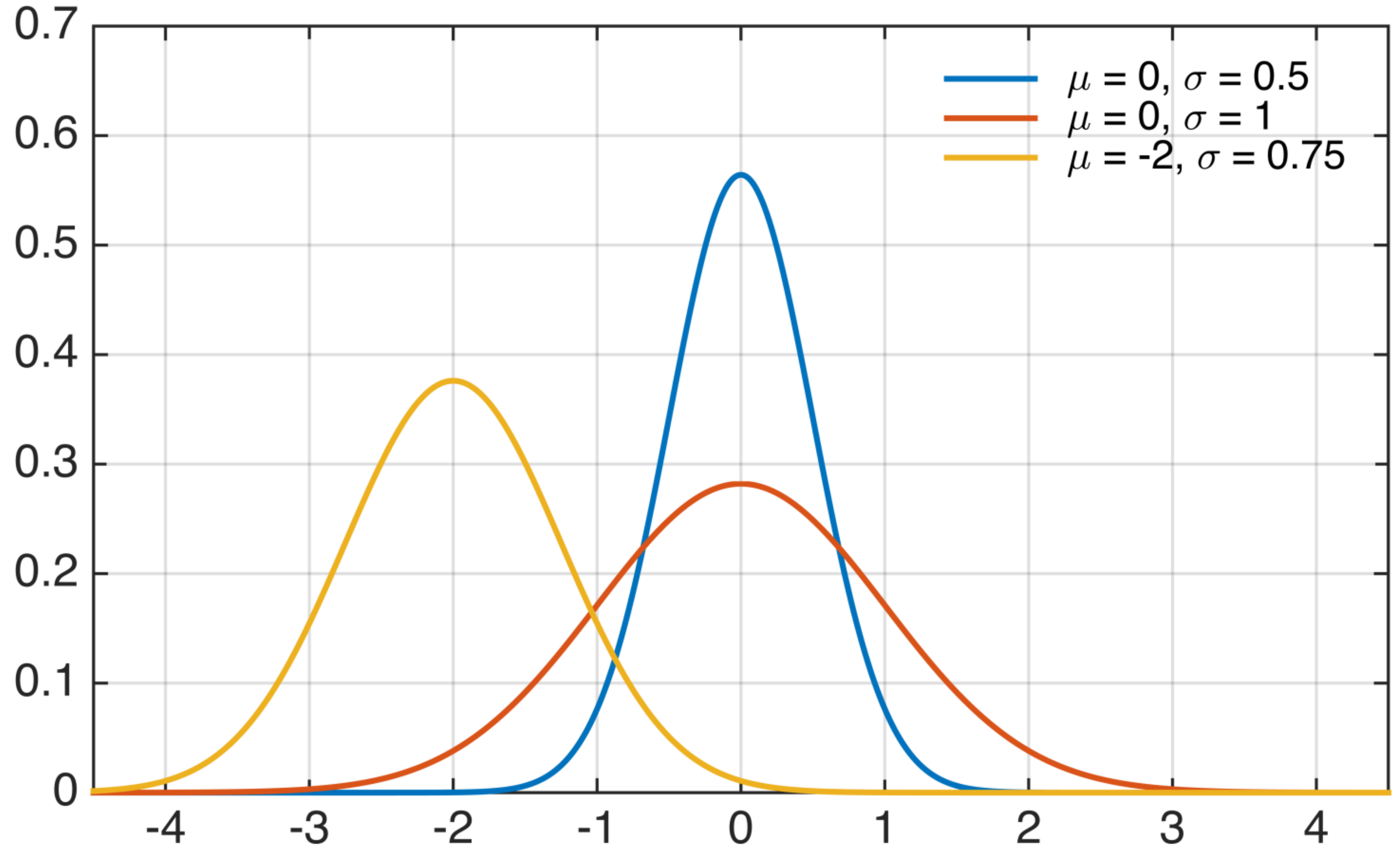


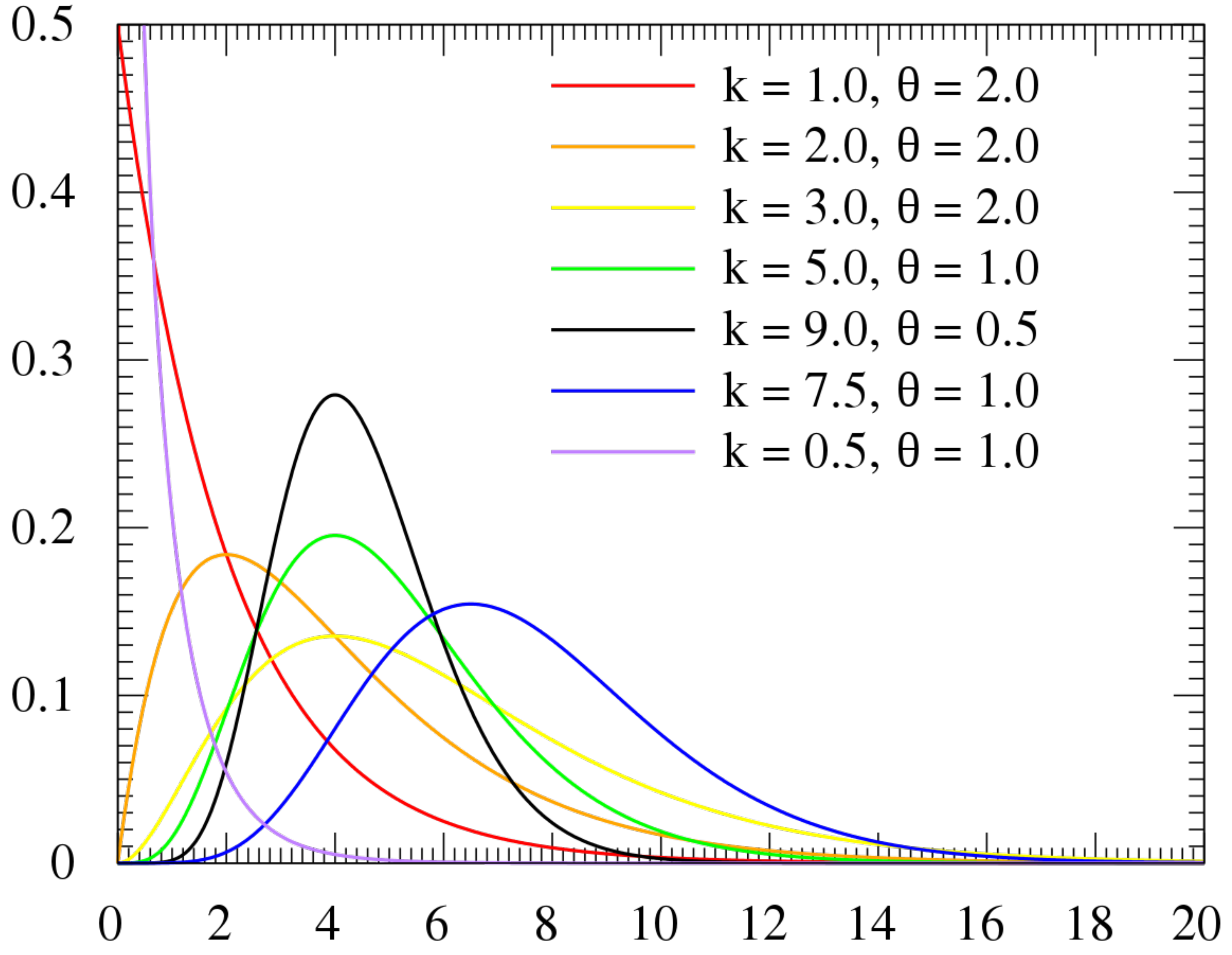


Exercise applying your knowledge

- A better choice is actually what is called a Gamma distribution

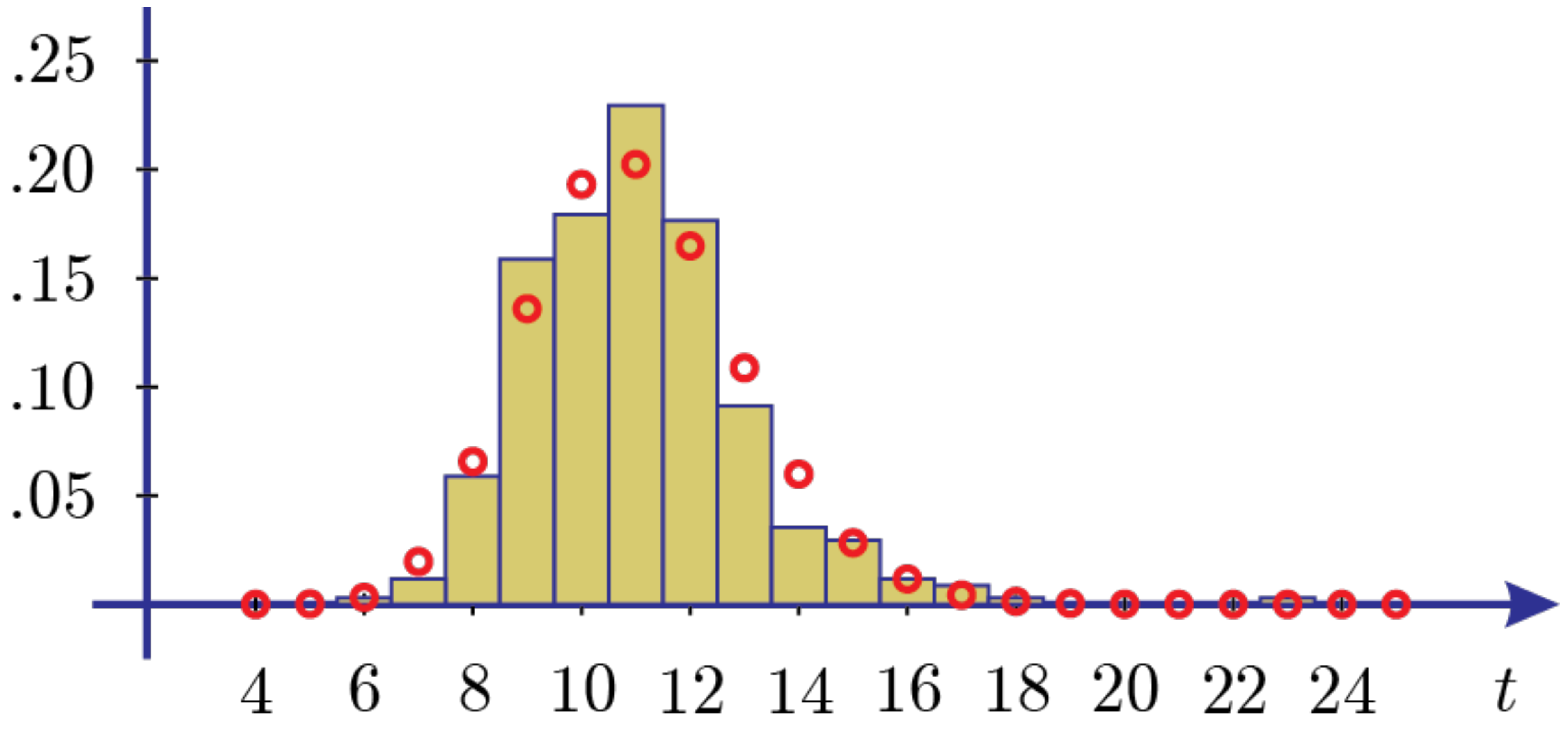






Exercise applying your knowledge

- We can also consider conditional distributions $p(y \mid x)$
- Y is the commute time, let X be the month
- Why is it useful to know $p(y \mid X = \text{Feb})$ and $p(y \mid X = \text{Sept})$?
- What else could we use for X and why pick it?

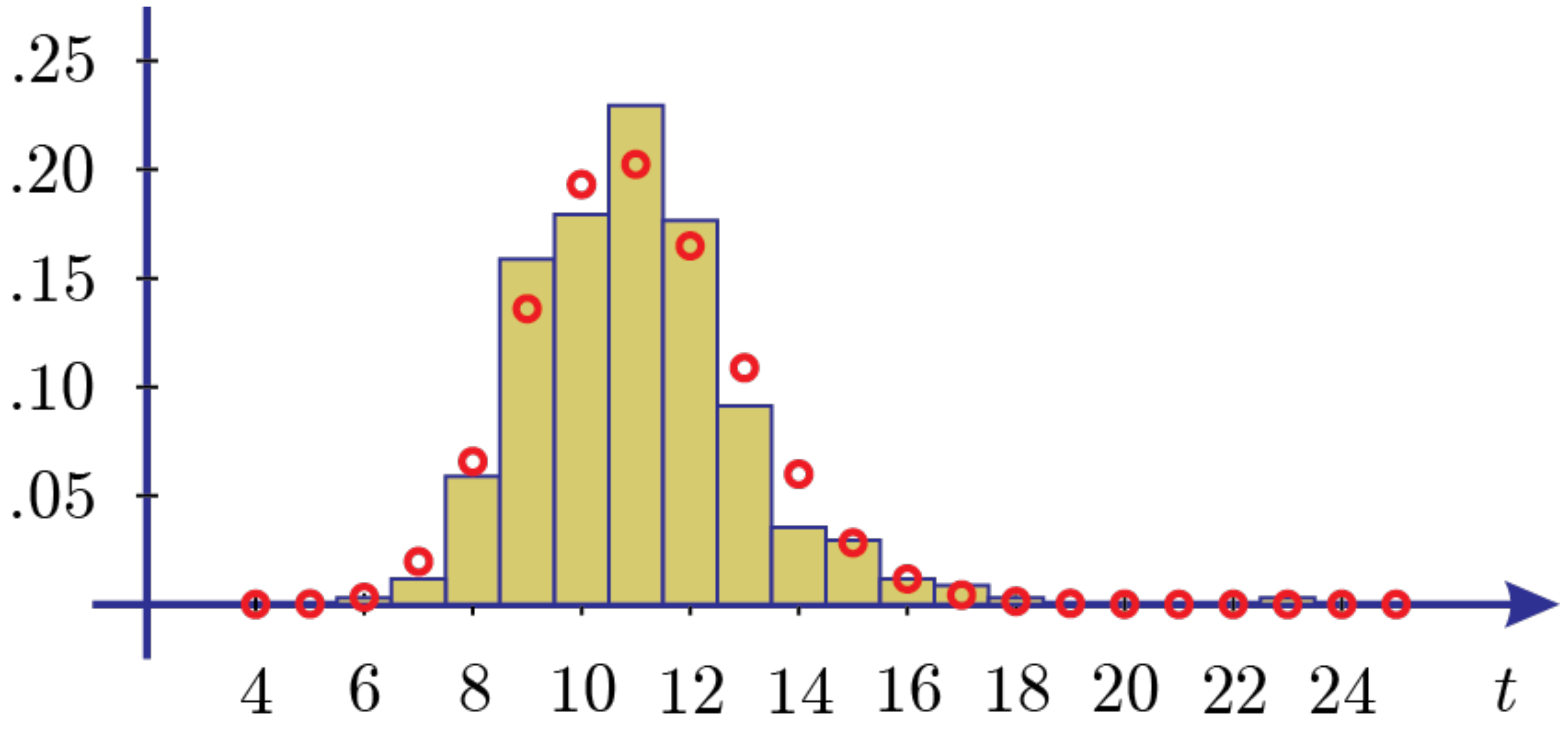


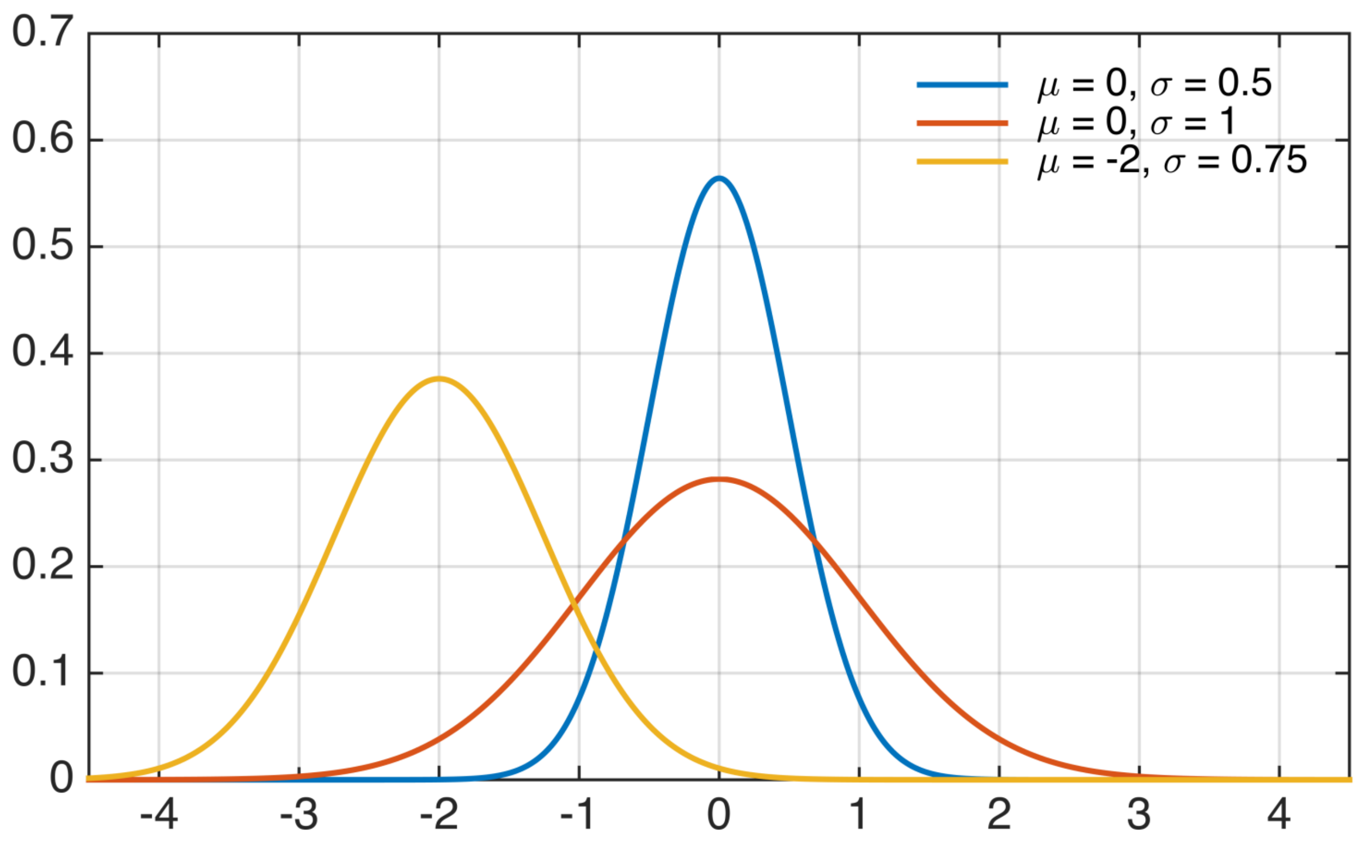
Exercise applying your knowledge

- Let's use a simple X, where it is 1 if it is slippery out and 0 otherwise
- Then we could model two Gaussians, one for the two types of conditions

$$p(y|X=0) = \mathcal{N}\left(\mu_0, \sigma_0^2\right)$$
$$p(y|X=1) = \mathcal{N}\left(\mu_1, \sigma_1^2\right)$$

Gaussian denoted by N





Exercise applying your knowledge

- Eventually we will see how to model the distribution over Y using functions of other variables (features) X

$$p(y|\mathbf{x}) = \mathcal{N}\left(\mu = \sum_{j=1}^{d} w_i x_i, \sigma^2\right)$$