Probability Theory

CMPUT 267: Basics of Machine Learning

§2.1-2.2

Recap for the Course Start

This class is about **understanding** machine learning techniques by understanding their basic **mathematical underpinnings**

- Please read the FAQ and Getting Started (it will save us all time)
- Assignment 1 released
- Readings Exercises due very soon (January 19)
  - Biggest reading since it covers much of the background
- Chapter 1 contains a mathematics refresher (sets, functions, derivatives)

Outline

1. Probabilities
2. Defining Distributions
3. Random Variables

Why Probabilities?

Even if the world is completely deterministic, outcomes can **look random**

Example: A high-tech gumball machine behaves according to
$f(x_1, x_2) =$ output candy if $x_1 \,\&\, x_2$,
where x_1 = **has candy** and x_2 = **battery charged**.

- You can only see if it has candy (only see x_1)
- From your perspective, when $x_1 = 1$, sometimes candy is output, sometimes it isn't
- It **looks stochastic**, because it depends on the hidden input x_2 (we only have partial observability)

Measuring Uncertainty

- **Probability** is a way of **measuring** uncertainty
- We assign a number between 0 and 1 to **events** (hypotheses):
  - **0** means absolutely certain that statement is **false**
  - **1** means absolutely certain that statement is **true**
  - **Intermediate** values mean more or less certain
- Probability is a measurement of **uncertainty**, **not truth**
  - A statement with probability .75 is not "mostly true"
  - Rather, we **believe** it is more **likely** to be true than not

Measuring Uncertainty and Gumballs

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- Probability is a measurement of **uncertainty**, **not truth**
  - A statement with probability .75 is not "mostly true"
  - Rather, we **believe** it is more **likely** to be true than not
- Gumball example: $f(x_1, x_2) =$ output candy if x_1 & x_2, where x_1 = **has candy** and x_2 = **battery charged**. We only observe x_1, and reason about the probability that a candy will be outputted (our belief about if it is likely to occur)

Another Example

- Let’s think about estimating the average height of a person in the world
- There is a true population mean h (say h = 165.2 cm)
  - which can be computed by averaging the heights of every person
- We can estimate this true mean using data
  - e.g., compute a sample average $\bar{h}$ from a subpopulation by randomly sampling 1000 people from around the whole world (say $\bar{h}$ = 166.3 cm)

Another Example
About uncertainty in our estimates

- Let's think about estimating the average height of a person in the world
- There is a true population mean h (say h = 165.2 cm)
- We can estimate this true mean using data
  - e.g., compute a sample average $\bar{h}$ from a subpopulation by randomly sampling 1000 people from around the whole world (say $\bar{h}$ = 166.3 cm)
- We can also reason about our belief over plausible estimates $\bar{h}$ of h
  - e.g., we can maintain a distribution over plausible $\bar{h}$, such as saying $p(\bar{h} = 160) = 0.1, p(\bar{h} = 163) = 0.3, p(\bar{h} = 165) = 0.5, p(\bar{h} = 167) = 0.1$

Now let’s get into formalizing all of this

Terminology Refresher

- Chapter 1 has a refresher and some exercises, and there is a notation sheet at the beginning of the notes
- Set notation
  - Curly brackets for discrete sets, e.g $\{a, b, c\}$, $\{1,2,3,4,5\}$, $\{-2.1,6.5\}$
  - Square brackets for continuous intervals, e.g., $[-10,10]$, $[3.2,7.1]$
  - Subset notation $A \subset \Omega$ and the set complement $A^c = \Omega \backslash A$
  - Union of sets $A \cup B$, intersection of sets $A \cap B$
  - Power set $\mathscr{P}(A)$, e.g, $A = \{1,2\}$, $\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
- Scalar $x \in \mathbb{R}$ and vector (array) is $\mathbf{x} \in \mathbb{R}^d$ for some integer $d \in \{2,3,\ldots\}$

Terminology (cont.)

- **Countable:** A set whose elements can be assigned an integer index
  - The integers themselves
  - Any finite set, e.g., $\{0.1, 2.0, 3.7, 4.123\}$
  - Usually we'll say we have a **discrete set**
- **Uncountable:** Sets whose elements *cannot* be assigned an integer index
  - Real numbers $\mathbb{R}$
  - Intervals of real numbers, e.g., $[0,1]$, $(-\infty, 0)$
  - Usually we'll say we have a **continuous set**

Outcomes and Events

All probabilities are defined with respect to a **measurable space** $(\Omega, \mathscr{E})$ of **outcomes** and **events**:

- Ω is the **sample space**: The set of all possible outcomes
- $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is the **event space**: A set of subsets of Ω that satisfies two key properties (that I will define in two slides)

Examples of Discrete & Continuous Sample Spaces and Events

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person, robot, camera, TV}, \ldots\}$

$\Omega = \mathbb{N}$

Continuous (uncountable) outcomes

$\Omega = [0,1]$

$\Omega = \mathbb{R}$

$\Omega = \mathbb{R}^k$

Event Spaces

Definition:

A non-empty set $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is an **event space** if it satisfies

1. $A \in \mathscr{E} \implies A^c \in \mathscr{E}$
2. $A_1, A_2, \ldots \in \mathscr{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$

1. A collection of outcomes (e.g., either a 2 or a 6 were rolled) is an event.
2. If we can measure that an event has occurred, then we should also be able to measure that the event has not occurred; i.e., its **complement** is measurable.
3. If we can measure two events separately, then we should be able to tell if one of them has happened; i.e., their **union** should be measurable too.

Examples of Discrete & Continuous Sample Spaces and Events

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person}, \text{robot}, \text{camera}, \text{TV}, \ldots\}$

$\Omega = \mathbb{N}$

$\mathscr{E} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\}$

Typically**:** $\mathscr{E} = \mathscr{P}(\Omega)$

Powerset is the set of all subsets

Continuous (uncountable) outcomes

$\Omega = [0,1]$

$\Omega = \mathbb{R}$

$\Omega = \mathbb{R}^k$

$\mathscr{E} = \{\emptyset, [0,0.5], (0.5,1.0], [0,1]\}$

Typically: $\mathscr{E} = B(\Omega)$ ("Borel field")

Borel field is the set of all subsets of
non-negligible size (e.g., intervals $[0.1, 0.1 + \epsilon]$)

Examples of Discrete & Continuous Sample Spaces and Events

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Typically: $\mathscr{E} = B(\Omega)$ ("Borel field")

Borel field is the set of all subsets of non-negligible size (e.g., intervals $[0.1,0.1 + \epsilon]$)

Note: *not* $\mathscr{P}(\Omega)$

Discrete vs. Continuous Sample Spaces

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person, robot, camera, TV}, \ldots\}$

$\Omega = \mathbb{N}$

$\mathscr{E} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\}$

Typically**:** $\mathscr{E} = \mathscr{P}(\Omega)$

Question:

$\mathscr{E} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$?

Definition:

A non-empty set $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is an **event space** if

1. $A \in \mathscr{E} \implies A^c \in \mathscr{E}$
2. $A_1, A_2, \ldots \in \mathscr{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$

Exercise

- Write down the power set of {1, 2, 3}
- More advanced: Why is the power set a valid event space? Hint: Check the two properties

Definition:

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Exercise answer

A set $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is an **event space** if it satisfies

1. $A \in \mathscr{E} \implies A^c \in \mathscr{E}$
2. $A_1, A_2, \ldots \in \mathscr{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$

- $\Omega = \{1,2,3\}$
- $\mathscr{P}(\Omega) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- Proof that the power set satisfies the two properties
- Take any $A \in \mathscr{P}(\Omega)$ (e.g., $A = \{1\}$ or $A = \{1,2\}$). Then $A^c = \Omega \backslash A$ is a subset of Ω, and so $A^c \in \mathscr{P}(\Omega)$ since the power set contains all subsets
- Take any $A, B \in \mathscr{P}(\Omega)$. Then $A \cup B \subset \Omega$, and so $A \cup B \in \mathscr{P}(\Omega)$
- More generally, for an infinite union, see: https://proofwiki.org/wiki/Power_Set_is_Closed_under_Countable_Unions

Axioms

Definition:

Given a measurable space $(\Omega, \mathscr{E})$, any function $P : \mathscr{E} \to [0,1]$ satisfying

1. unit measure: $P(\Omega) = 1$, and
2. σ-additivity: $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ for any countable sequence $A_1, A_2, \ldots \in \mathscr{E}$ where $A_i \cap A_j = \varnothing$ whenever $i \neq j$

is a **probability measure** (or **probability distribution**).

Defining a Distribution

Example:

$$\Omega = \{0,1\}$$

$$\mathscr{E} = \{\varnothing, \{0\}, \{1\}, \Omega\}$$

$$P = \begin{cases} 1 - \alpha & \text{if } A = \{0\} \\ \alpha & \text{if } A = \{1\} \\ 0 & \text{if } A = \varnothing \\ 1 & \text{if } A = \Omega \end{cases}$$

where $\alpha \in [0,1]$.

Questions:

1. Do you recognize this distribution?
2. How should we choose P in practice?
   a. Can we choose an arbitrary function?
   b. How can we guarantee that all of the constraints will be satisfied?

We will define distributions using **PMFs** and **PDFs**

Probability Mass Functions (PMFs)

Definition: Given a **discrete** sample space Ω, any function $p : \Omega \to [0,1]$ satisfying $\sum_{\omega \in \Omega} p(\omega) = 1$ is a **probability mass function**.

- For a discrete sample space, instead of defining P directly, we can define a **probability mass function** $p : \Omega \to [0,1]$.
- p gives a probability for **outcomes** instead of **events**
- The probability for any event $A \in \mathscr{E}$ is then defined as $P(A) = \sum_{\omega \in A} p(\omega)$.

Example: PMF for a Fair Die

A **categorical distribution** is a distribution over a **finite** outcome space, where the probability of each outcome is specified separately.

Example: Fair Die

$$\Omega = \{1,2,3,4,5,6\}$$

$$p(\omega) = \frac{1}{6}$$

ω	$p(\omega)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Questions:

1. What is a possible event? What is its probability?
2. What is the event space?

Example: PMF for a Fair Die

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Example: Fair Die

$$\Omega = \{1,2,3,4,5,6\}$$

$$p(\omega) = \frac{1}{6}$$

$$P(\{3,4\}) = \frac{1}{3}$$

ω	$p(\omega)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Questions:

1. What is a possible event? What is its probability?
2. What is the event space?

Moving to Boolean Terminology with Random Variables

Fair Die: $\Omega = \{1,2,3,4,5,6\}$, PMF $p(\omega) = \frac{1}{6}$

Instead of writing $P(\{3,4\}) = \frac{1}{3}$ with event $\{3,4\}$, more convenient to write

$P(X \in \{3,4\}) = \frac{1}{3}$ or $P(3 \leq X \leq 4) = \frac{1}{3}$

where X = the outcome of the die (the random variable)

Wait, why is it useful to move to RVs and Boolean terminology?

Example: Suppose we observe both a die's number, and where it lands.

$$\Omega = \{(left,1), (right,1), (left,2), (right,2), \ldots, (right,6)\}$$

We might want to think about the probability that we get a large number, without thinking about where it landed.

Let X = number that comes up. We could ask about $P(X = 3)$ or $P(X \geq 4)$

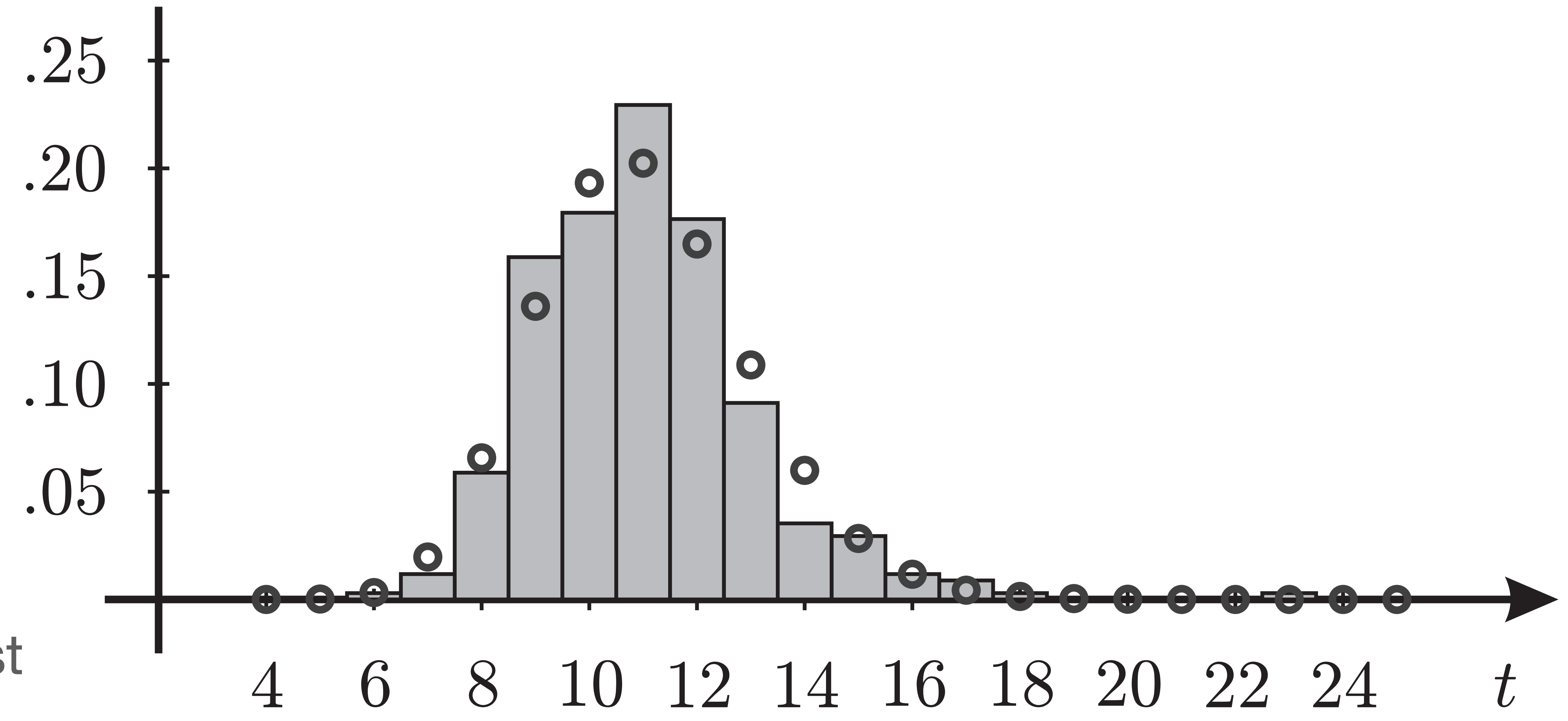
This is simpler to write than using the event notation, e.g,

$P(X = 3)$ would be written $P(\{\omega \in \Omega \mid \omega_2 = 3\})$

We will now always reason about probabilities using random variables
All the exact same probability rules apply (just rewritten using booleans)

Example: Using a PMF

- Suppose that you recorded your commute time (in minutes) every day for a year (i.e., 365 recorded times). (this is your dataset)
- Random variable $T =$ the commute time, with outcomes $\{4,5,6,7,\ldots,25\}$
- **Question:** How do you get this pmf from the data?
-



Note: Ignore the dots, they are obtained learning a gamma dist

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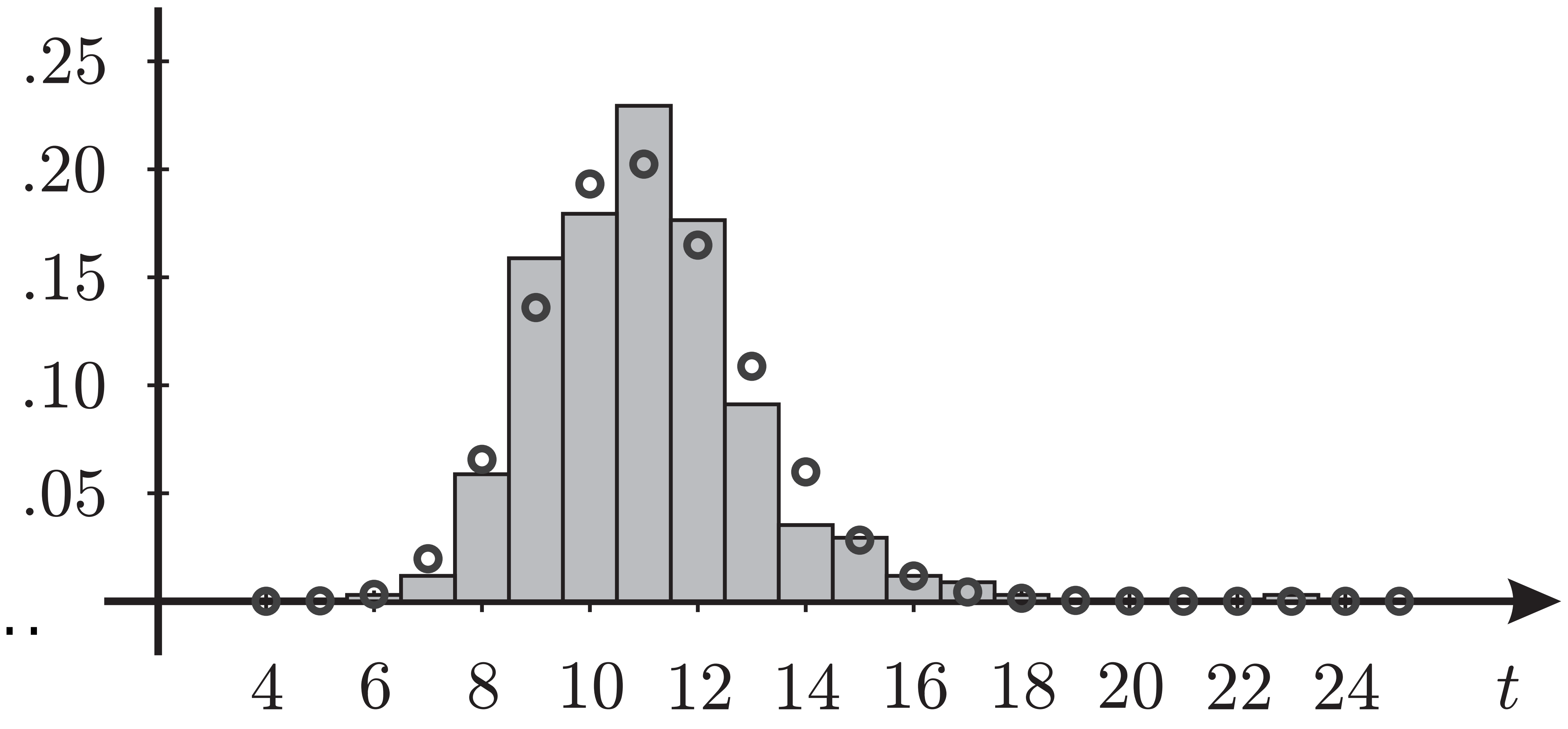
- Random variable $T =$ the commute time, with outcomes $\{4,5,6,7,\ldots,25\}$
- **Question:** How do you get this pmf from the data?
- Make a histogram and normalize
- Count the number of times you have seen a 4, 5, 6, …, 25
- Normalize by 365, to get the proportion of the time you saw 4, 5, …
- Example: #4s = 3, #5s = 4, #6s = 4, …
p(4) = 3/365 = 0.008, p(5) = 4/365 = 0.01, …



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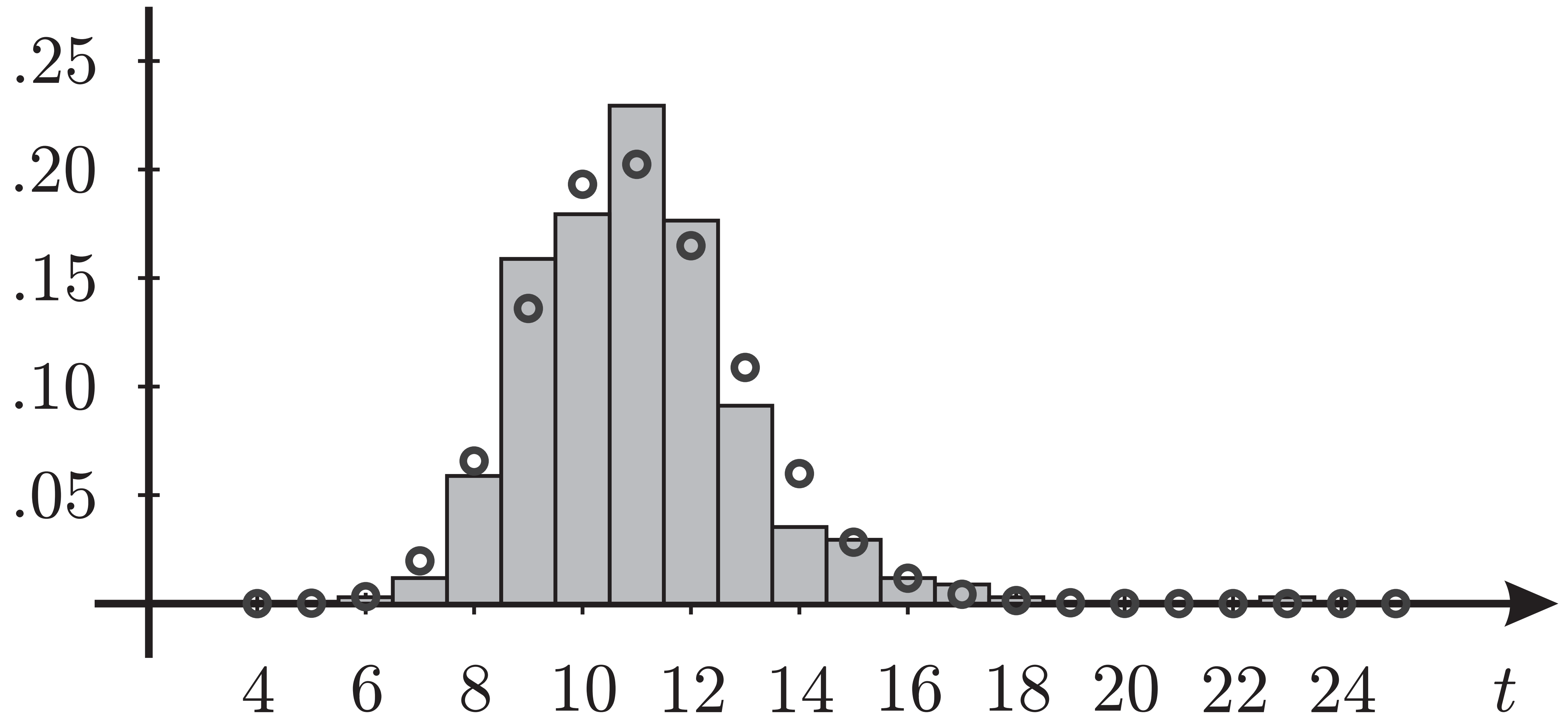

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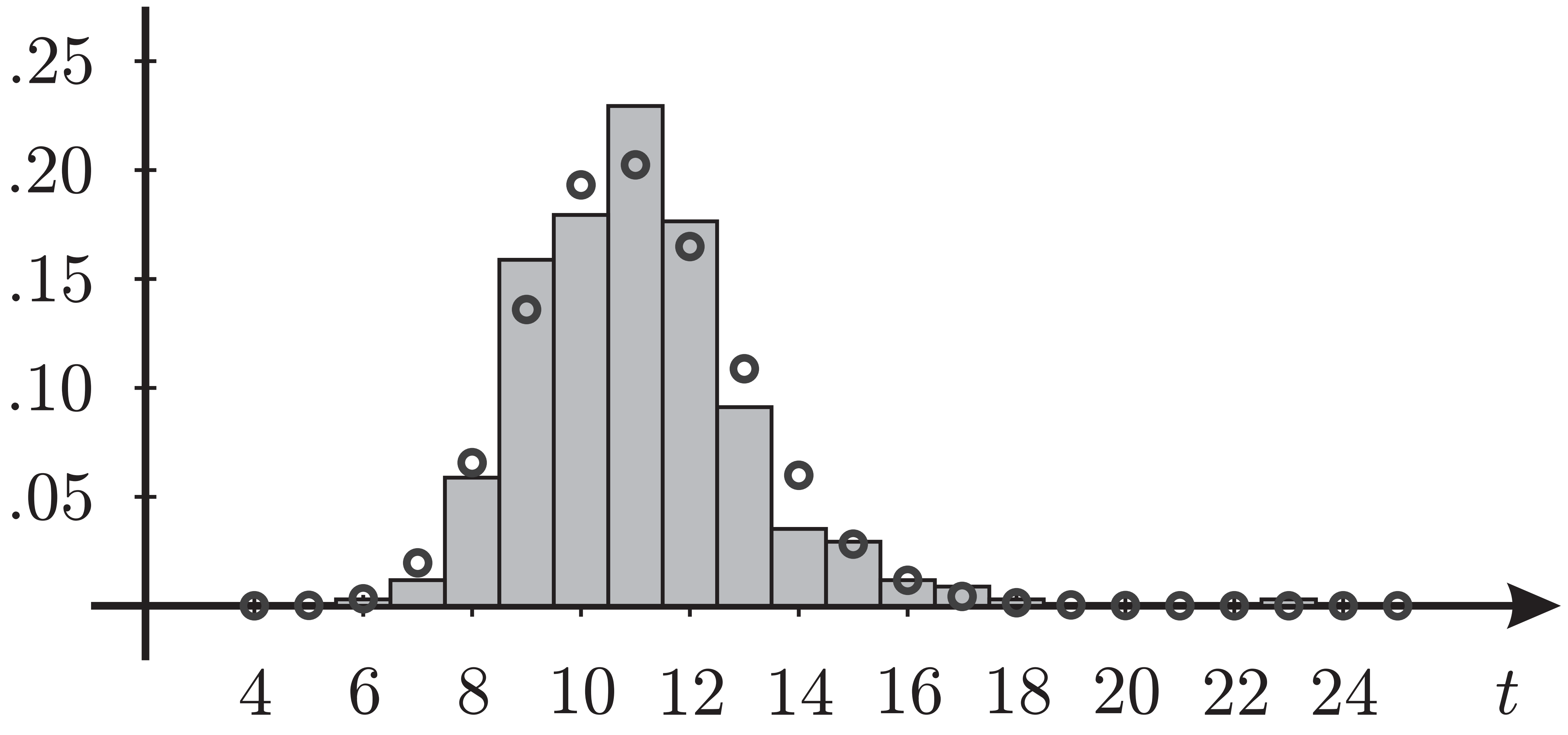
- **Question:** How do you get this pmf from the data?
- **Question:** How can you use this pmf to make predictions?
- Mode = most likely outcome
- Here that is 11. A reasonable prediction for your commute time (based only on p) is the mode of p



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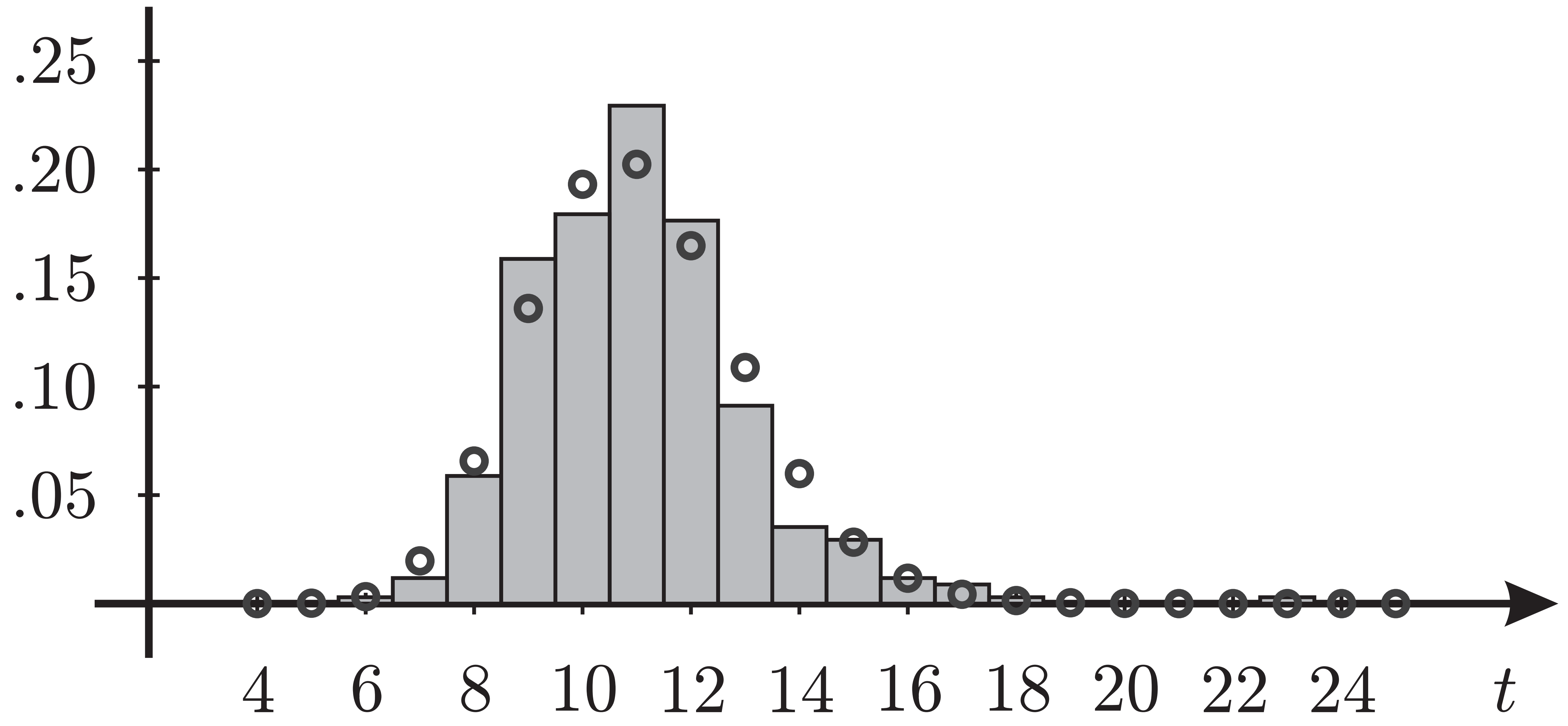
- **Question:** How do you get this pmf from the data?
- **Question:** How can you use this pmf to make predictions?
- **Question:** How do you compute $P(10 \leq T \leq 13)$?



To help you answer

Definition: Given a discrete sample space Ω, any function $p : \Omega \rightarrow [0,1]$ satisfying $\sum_{\omega \in \Omega} p(\omega) = 1$ is a **probability mass function**.

- The PMF defines the distribution for the random variable T
- $P(T \in \mathscr{A}) = \sum_{t \in A} p(t)$.

Example: Using a PMF

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- Random variable $T =$ the commute time, with outcomes $\{4,5,6,7,\ldots,25\}$
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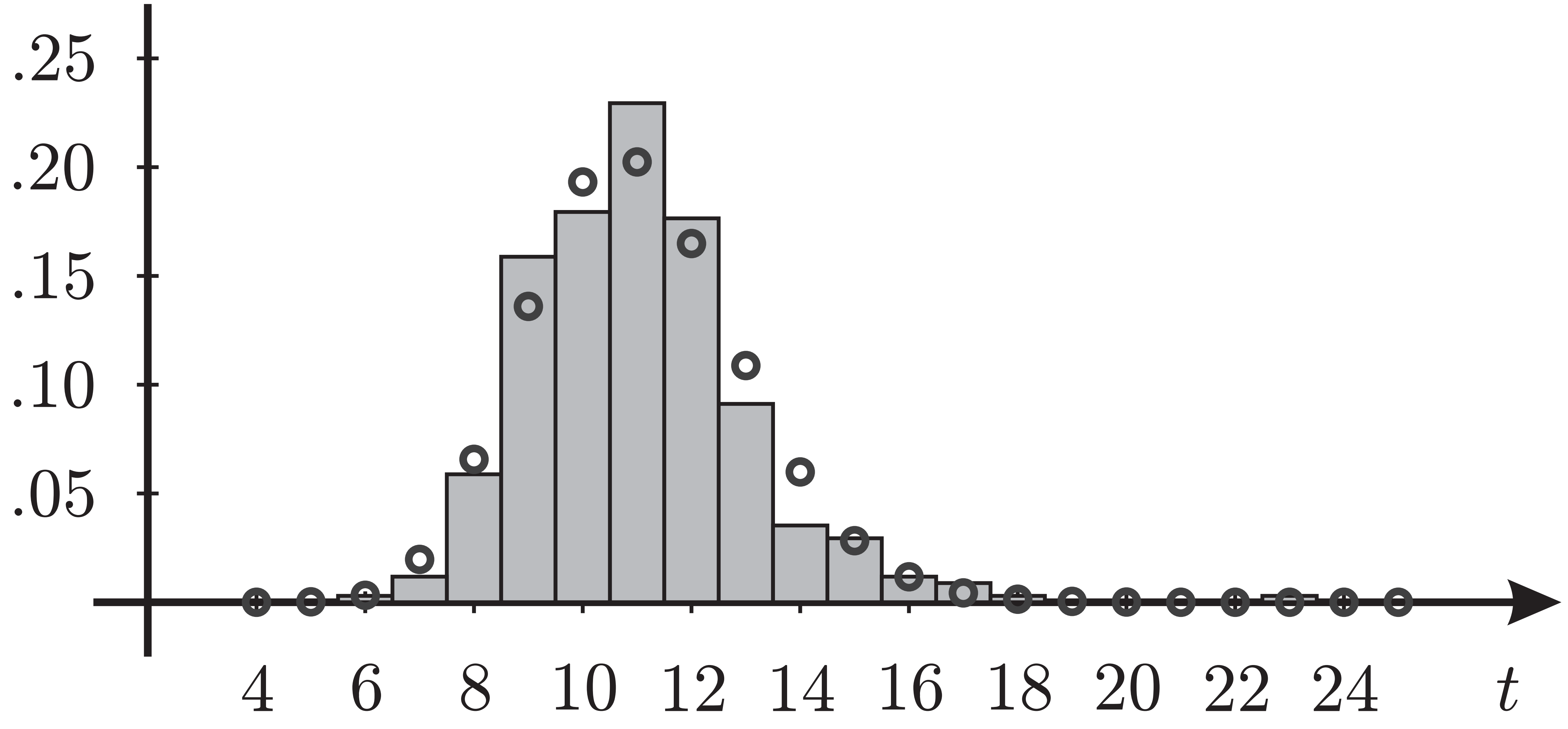
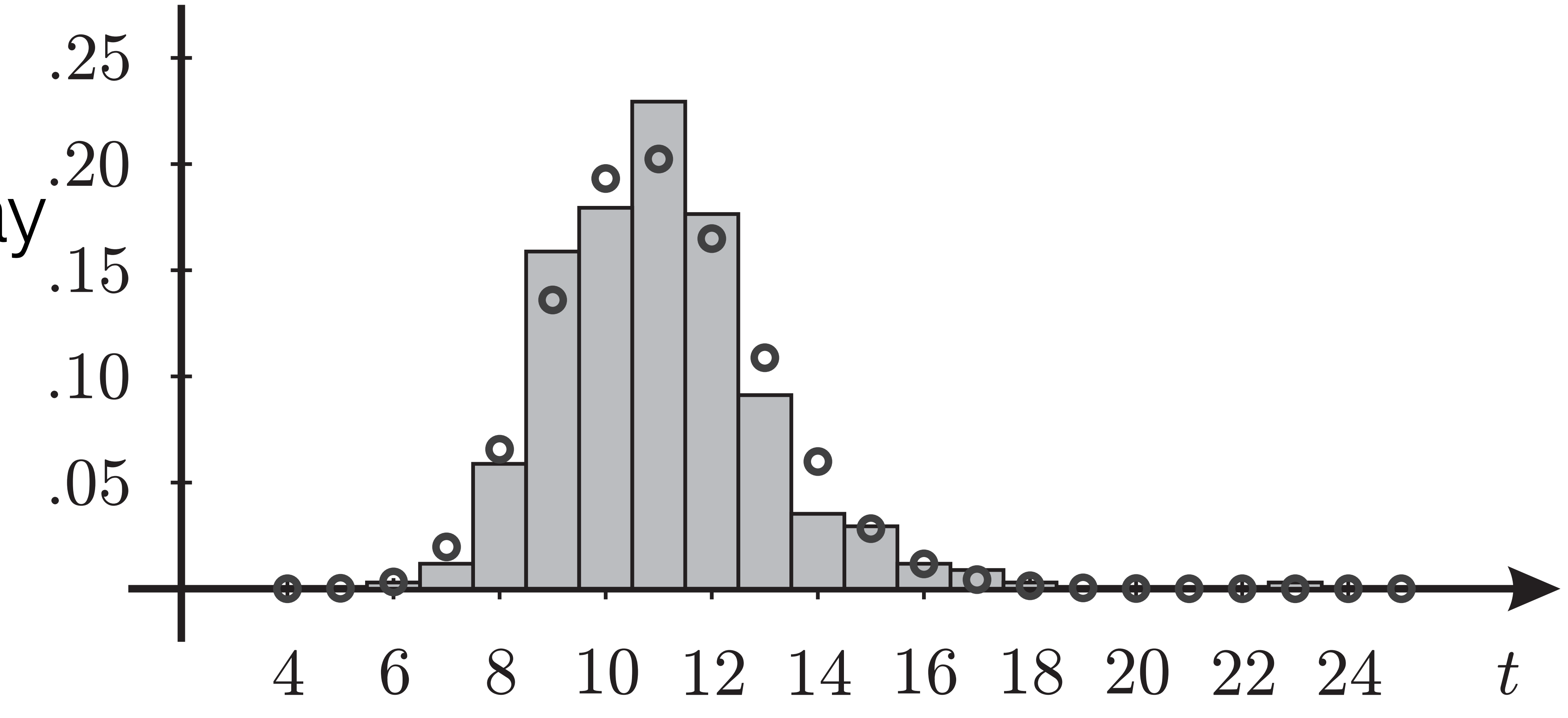
- **Question:** How can you use this pmf to make predictions?
- **Question:** How do you compute $P(10 \leq T \leq 13)$?
  Hint: $P(T \in \mathscr{A}) = \sum_{t \in A} p(t)$



Example: Using a PMF

- Suppose that you recorded your commute time (in minutes) every day for a year (i.e., 365 recorded times).
- Random variable $T =$ the commute time, with outcomes $\{4,5,6,7,\ldots,25\}$
- **Question:** How do you get $p(t)$? (Answer: count and normalize)
- **Question:** How is $p(t)$ useful?
  - We can take mode as prediction, and see the likelihood of different commute times for today
- **Question:** How do you compute $P(10 \leq T \leq 13)$?
  - Answer: $\sum_{t \in \{10,11,12,13\}} p(t)$



This PMF is called a categorical distribution, with 21 categories (table of probabilities)

PMFs are usually probability tables

- If you have m discrete outcomes (e.g., m = 5 or m = 10 or m = 43), then we just need a table of probabilities for the pmf
    - Q: how do we know this pmf give by this table here is valid?

Outcome	**1**	**2**	**3**	**4**	**5**
p(x)	0.1	0.25	0.02	0.4	0.23

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- Unless we have infinitely many discrete outcomes
- And we also name the Bernoulli (coin flip) distribution since we use it so much

Useful PMFs: Bernoulli

A **Bernoulli distribution** is a special case of a **categorical distribution** in which there are only two outcomes. It has a single **parameter** $\alpha \in (0,1)$.

$\mathcal{X} = \{T, F\}$ (or $\mathcal{X} = \{S, F\}$)

$$p(x) = \begin{cases} \alpha & \text{if } x = T \\ 1 - \alpha & \text{if } x = F \end{cases}.$$

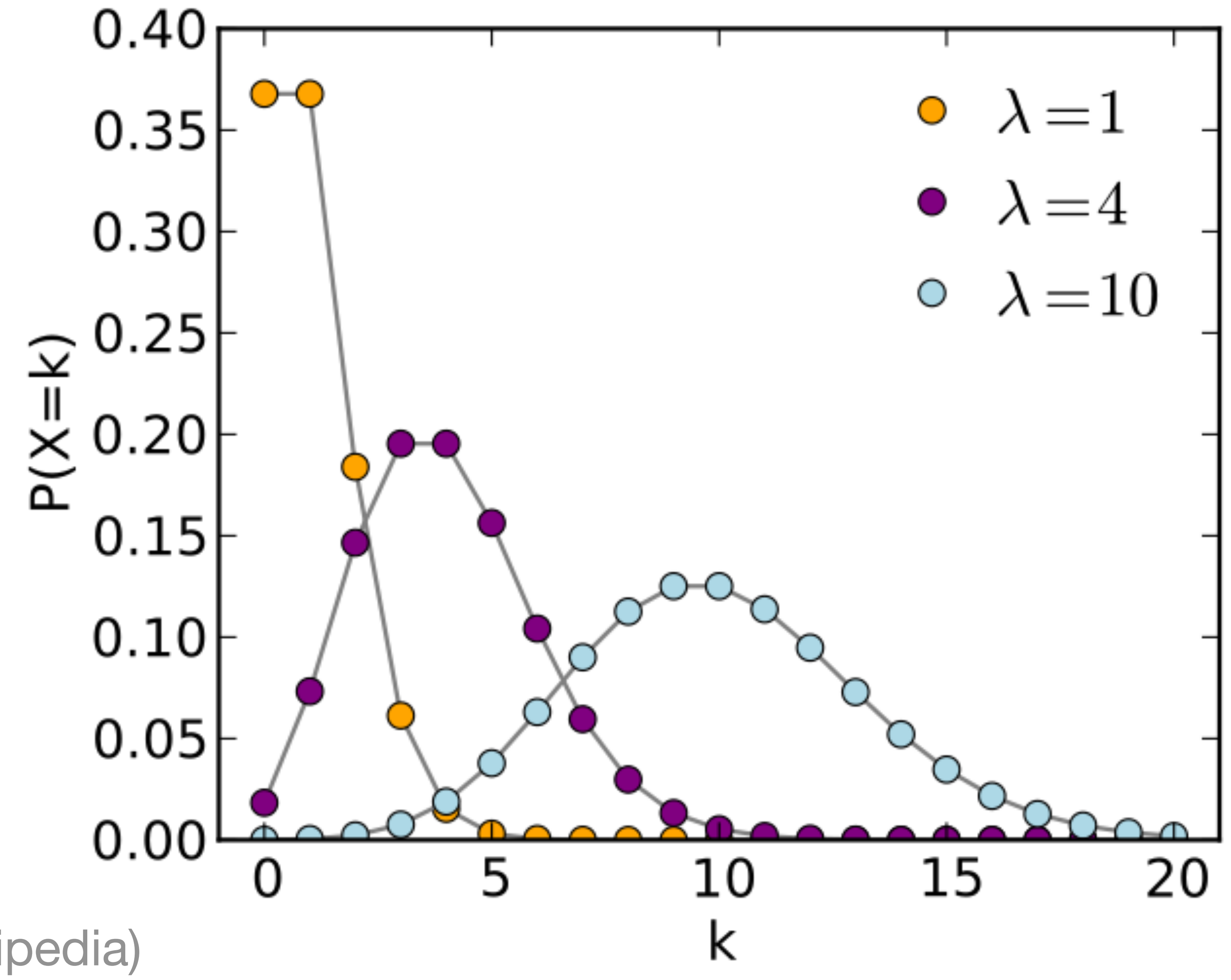
Alternatively: $\mathcal{X} = \{0,1\}$

$$p(x) = \alpha^x (1 - \alpha)^{1-x} \text{ for } x \in \{0,1\}$$

Useful PMFs: Poisson

A **Poisson distribution** is a distribution over the non-negative integers.
It has a single parameter $\lambda \in (0,\infty)$.

E.g., number of calls received by a call centre in an hour, λ is the average number of calls



(Image: Wikipedia)

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Questions:

1. Could we define this with a table instead of an equation?
2. How can we check whether this is a valid PMF?
3. λ real-valued, but outcomes are discrete. What might be the mode (most likely outcome)?

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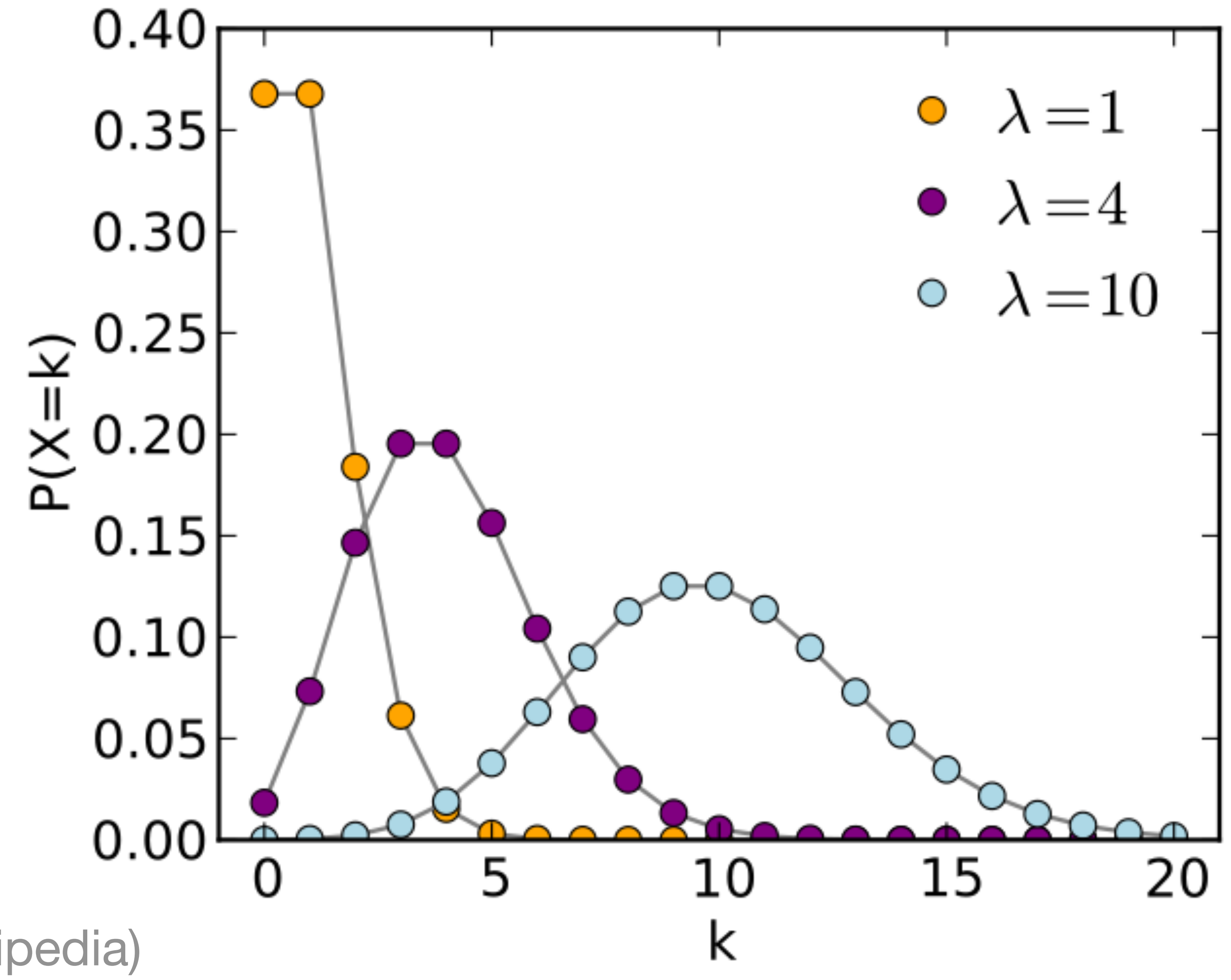


(Image: Wikipedia)

1. Could we define this with a table instead of an equation?
   - No because the outcome space is infinite

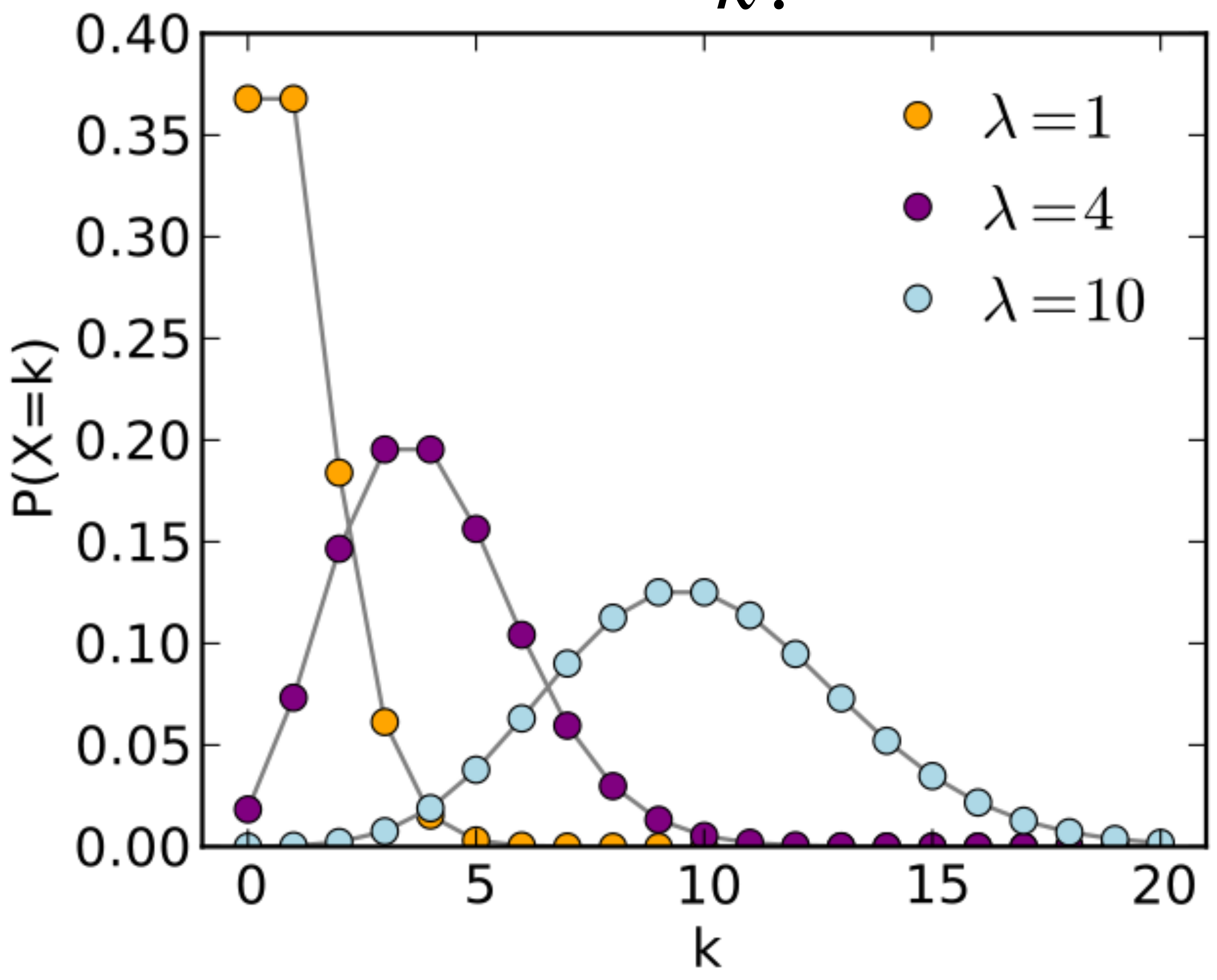
2. How can we check whether this is a valid PMF?
   - Check if $\sum_{k=0}^{\infty} p(k) = 1$

3. λ real-valued, but outcomes are discrete. What might be the mode (most likely outcome)?
   - Mean is λ, may not correspond to any outcome
   - Two modes, $\lceil \lambda \rceil - 1, \lfloor \lambda \rfloor$

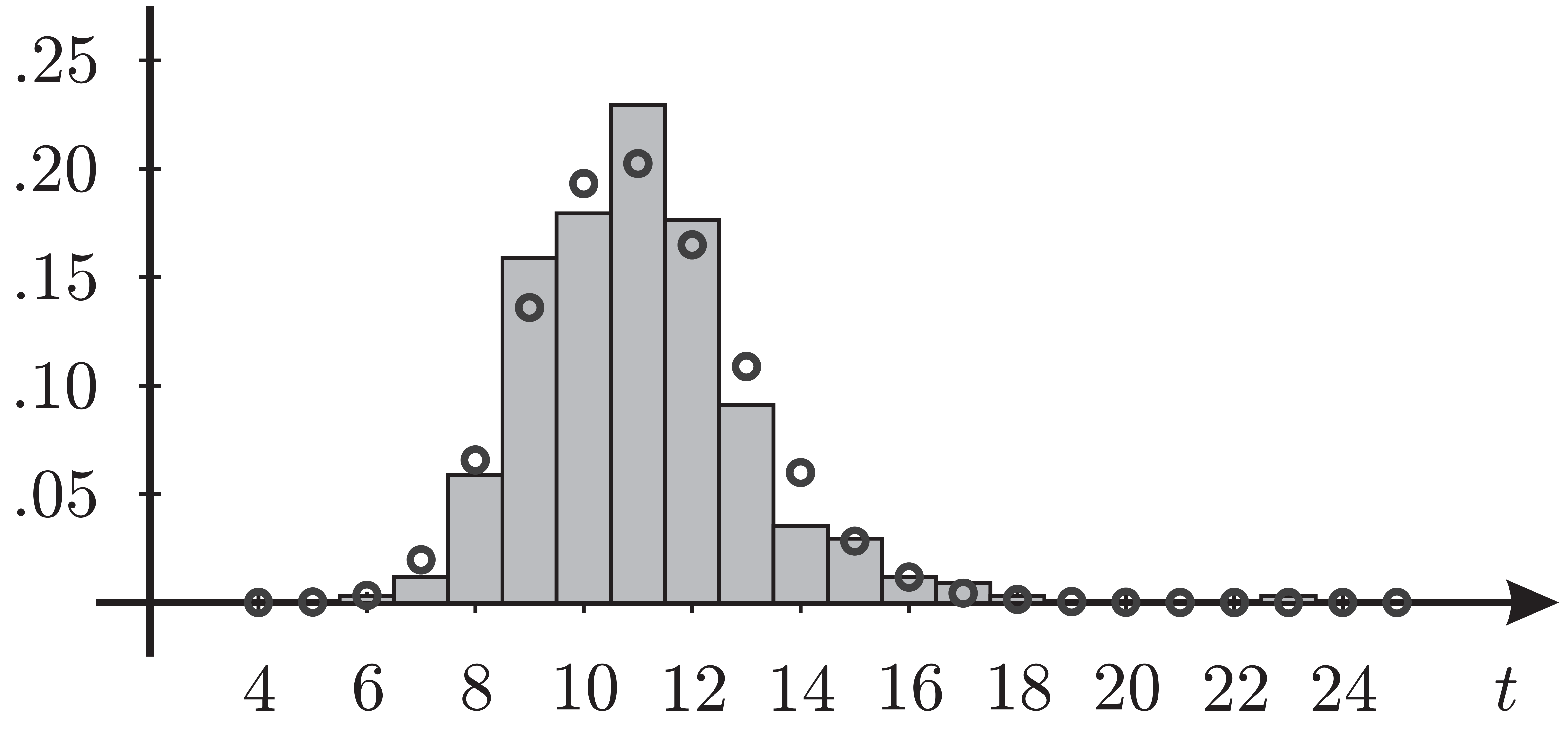
Commute Times Again

- **Question:** Could we use a **Poisson distribution** for commute times (instead of a categorical distribution)?
- **Question:** What would be the benefit of using a Poisson distribution? Hint: what do you need to estimate to specify the Poisson, vs the categorical?

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

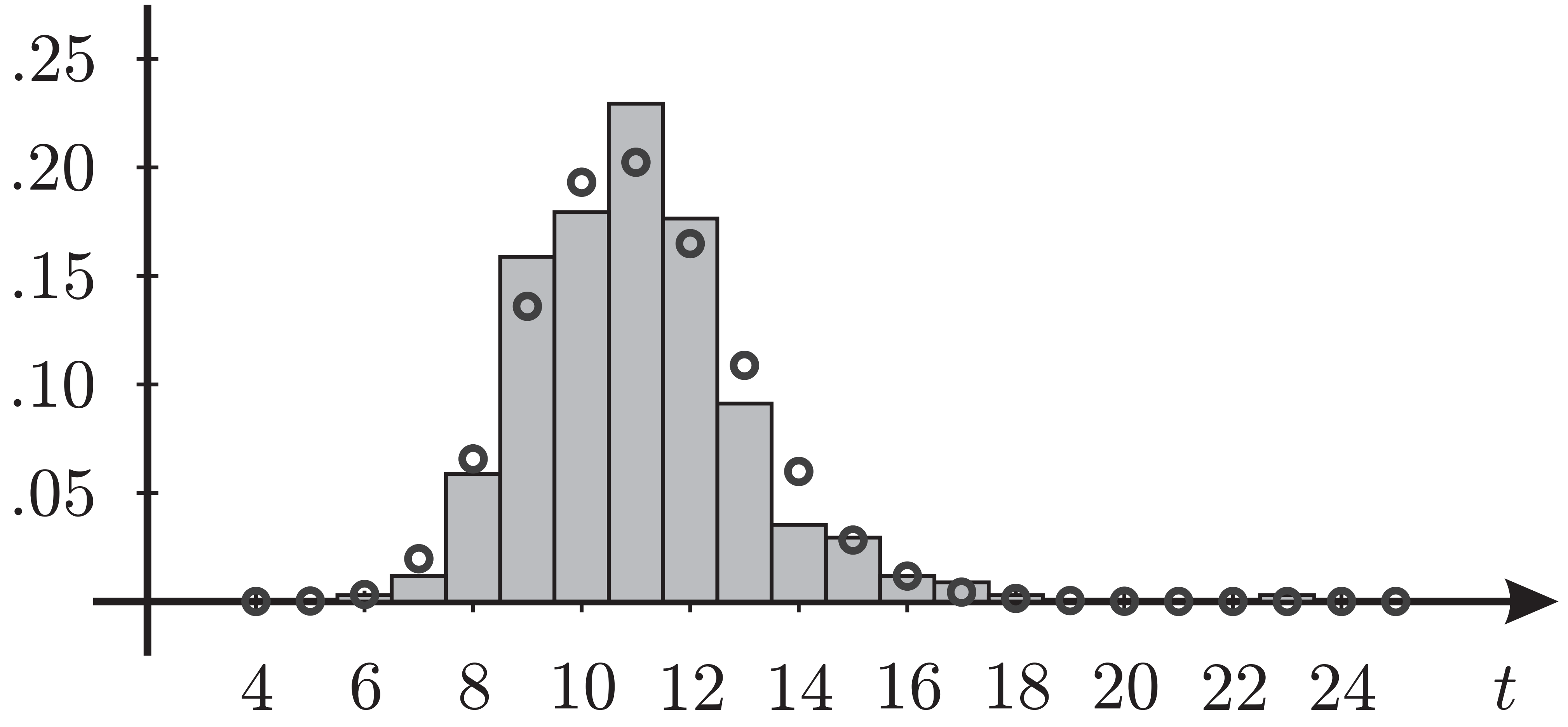


$$p(4) = 1/365,\ p(5) = 2/365,\ p(6) = 4/365,\ \ldots$$



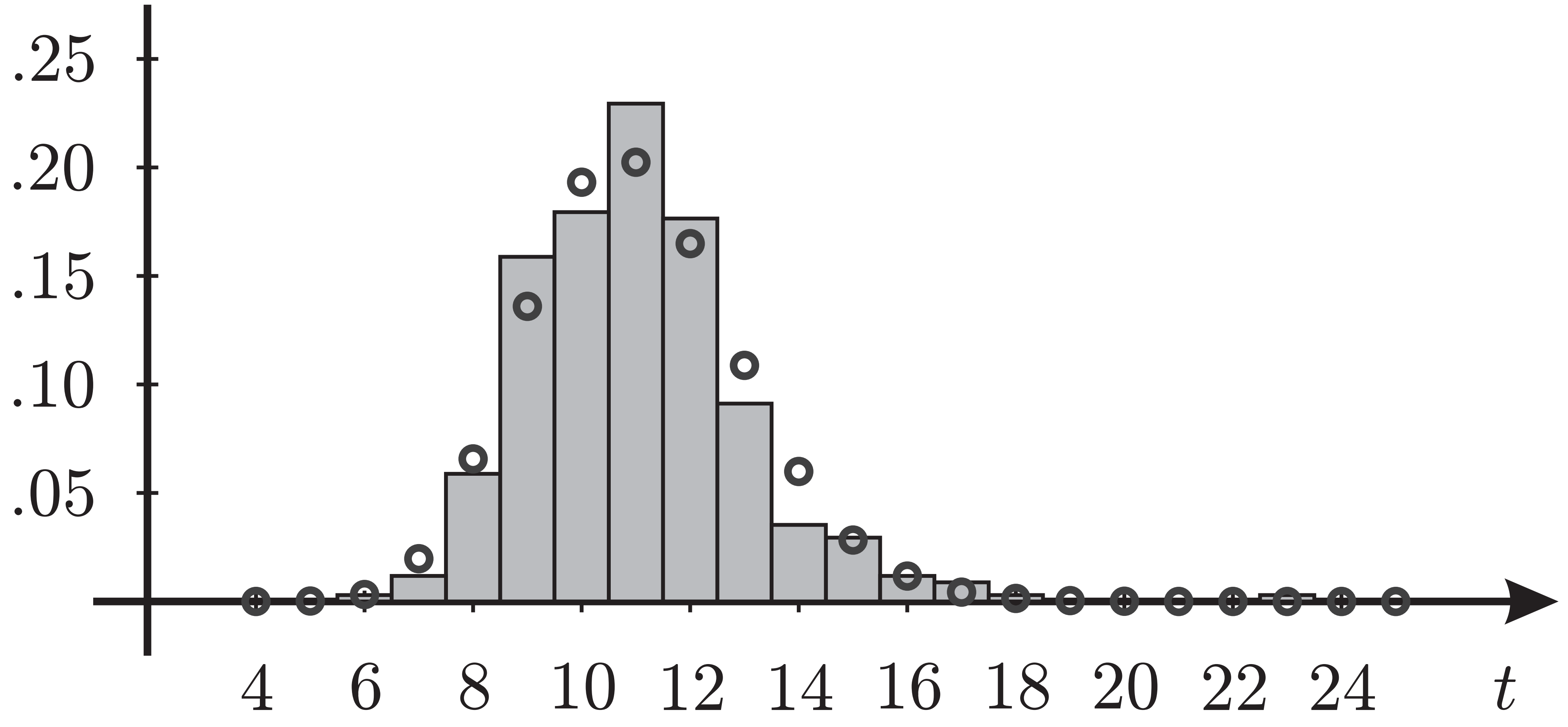
Continuous Commute Times

- It never actually takes *exactly* 12 minutes; I rounded each observation to the nearest integer number of minutes.
  - Actual data was 12.345 minutes, 11.78213 minutes, etc.



Continuous Commute Times

- It never actually takes *exactly* 12 minutes; I rounded each observation to the nearest integer number of minutes.
  - Actual data was 12.345 minutes, 11.78213 minutes, etc.
- **Question:** Could we use a Poisson distribution to predict the *exact* commute time (rather than the nearest number of minutes)? Why?



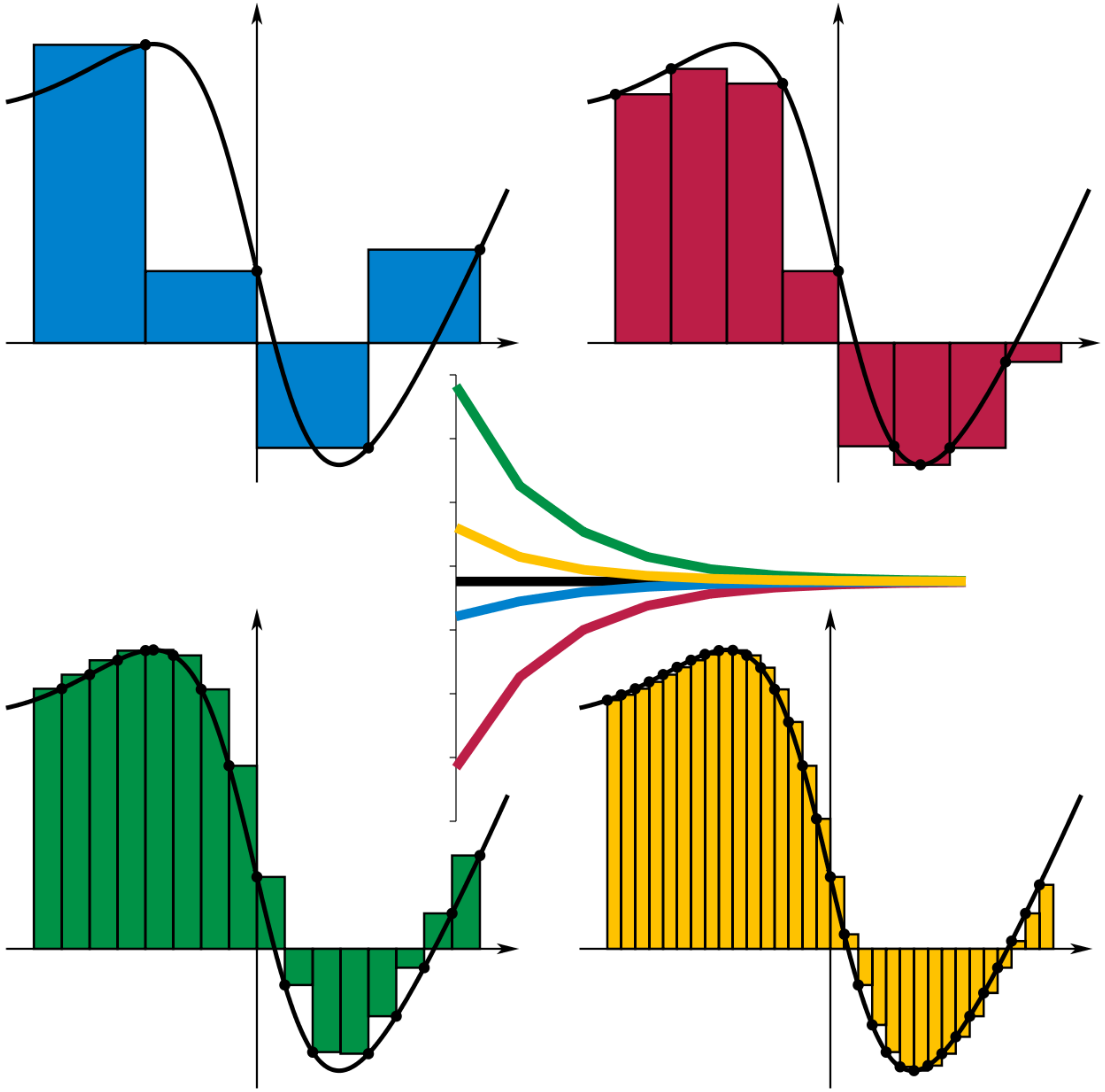
Probability Density Functions (PDFs)

Definition: Given a continuous sample space $\mathscr{X}$, any function $p : \mathscr{X} \to [0,\infty)$ satisfying $\int_{\mathscr{X}} p(x)dx = 1$ is a **probability density function**.

- For a continuous sample space, instead of defining P directly, we can define a **probability density function** $p : \mathscr{X} \to [0,\infty)$.
- The probability for any event $A \in \mathscr{E}$ is defined as

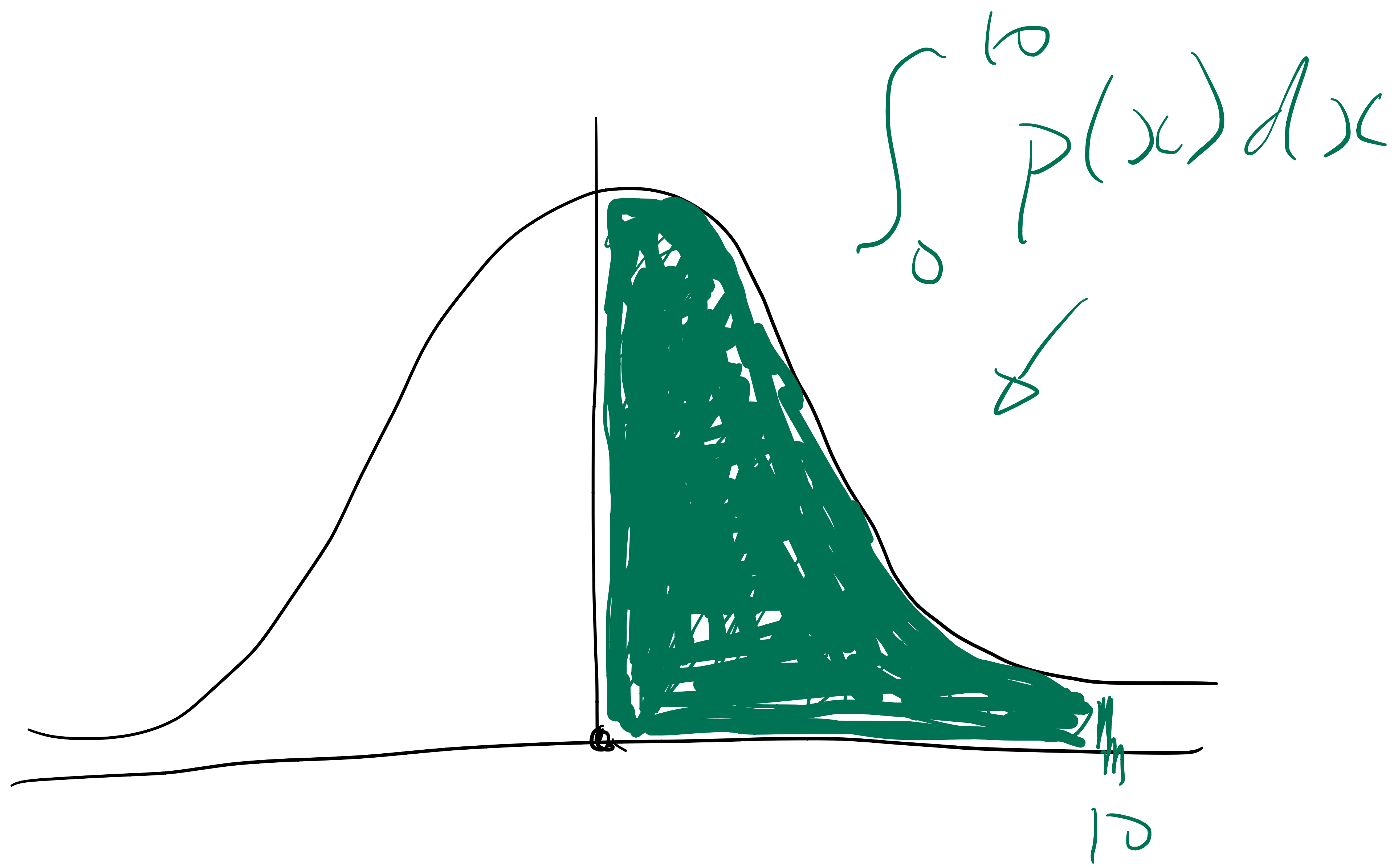
$$P(X \in A) = \int_A p(x)dx.$$

Recall Integration



Integration to give the probability of an event

- Imagine the PDF looks like the following concave function



$$P(0 \le X \le 10) = \int_0^{10} p(x)dx$$

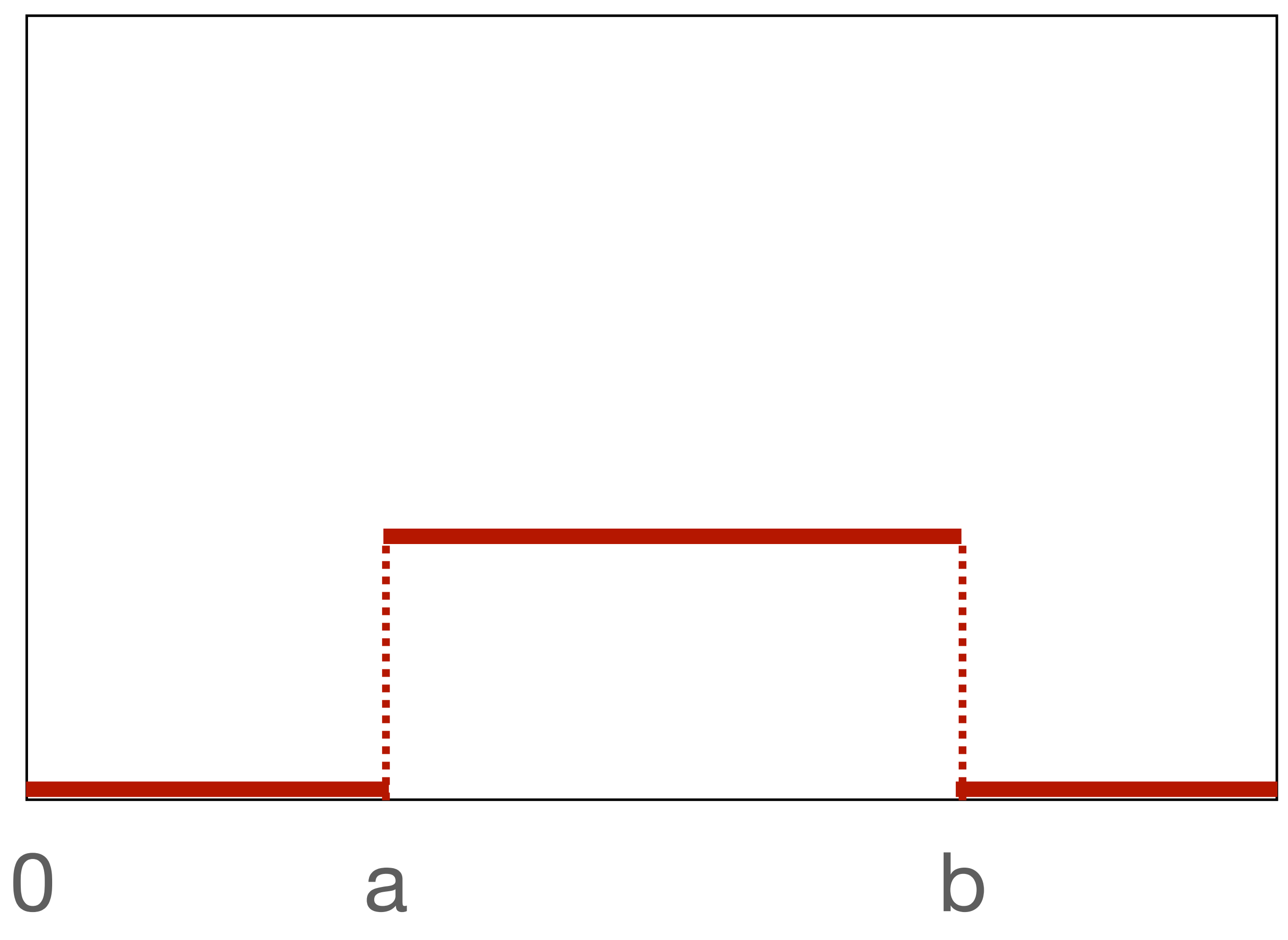
Area under the curve reflects the probability of seeing an outcome in that region

Useful PDFs: Uniform

A **uniform distribution** is a distribution over a real interval. It has two parameters: a and b.

$$\mathcal{X} = [a, b]$$

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$



Question: Does $\mathcal{X}$ have to be bounded?

Exercise: Check that the uniform pdf satisfies the required properties

Recall that the antiderivative of 1 is x, because the derivative of x is 1

$$\int_a^b p(x)dx = \int_a^b \frac{1}{b-a}dx$$

$$= \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}x\,|_a^b$$

$$= \frac{1}{b-a}(b-a) = 1$$

Useful PDFs: Gaussian

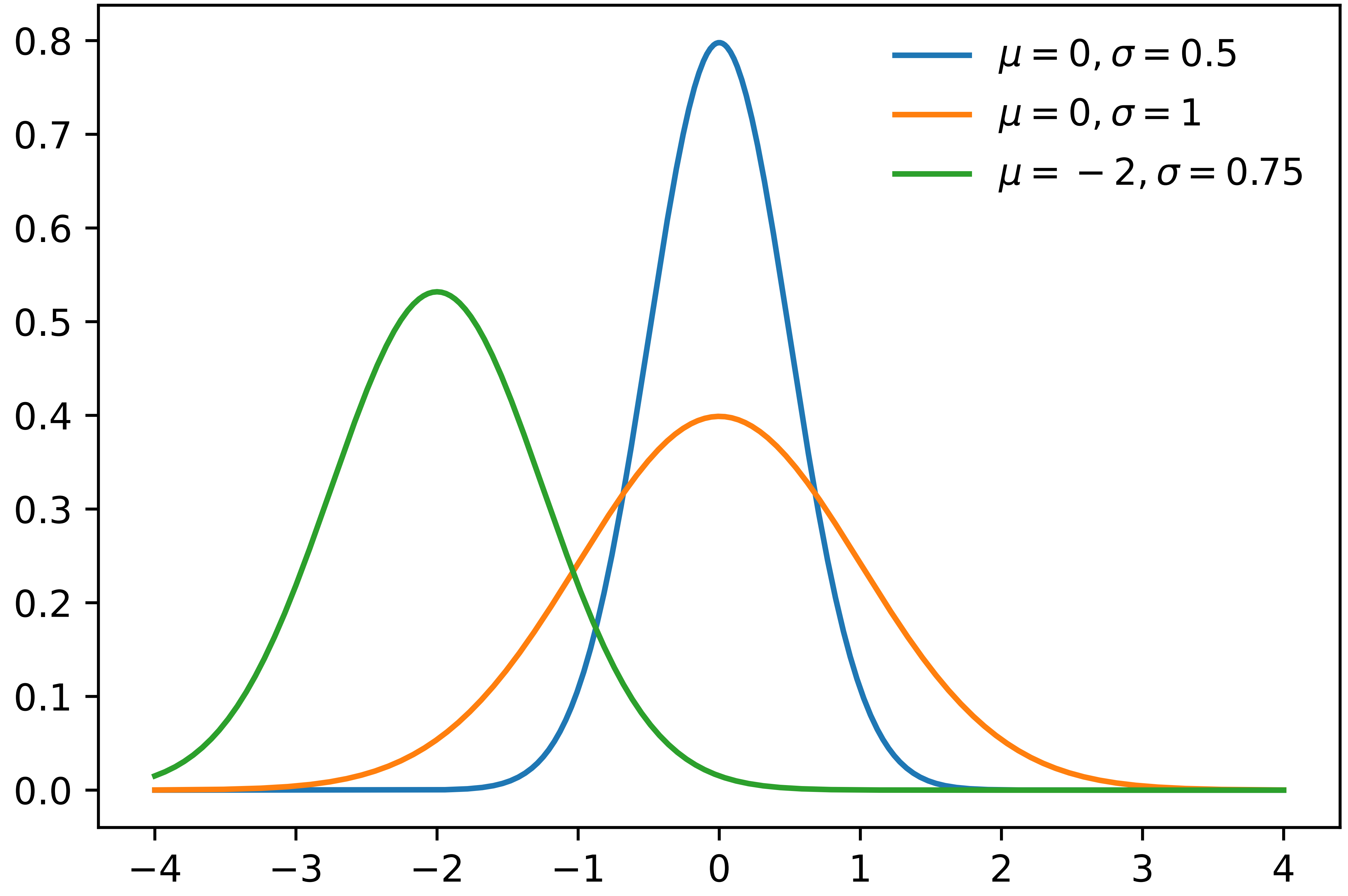
A **Gaussian distribution** is a distribution over the real numbers. It has two parameters: $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$.

$\mathcal{X} = \mathbb{R}$

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

where $\exp(x) = e^x$

Also called a normal distribution and written $\mathcal{N}(\mu, \sigma^2)$



Why the distinction between PMFs and PDFs?

1. When the sample space $\mathcal{X}$ is **discrete**:
   - Singleton event: $P(X \in \{x\}) = p(x)$ for $x \in \mathcal{X}$

$$P(A) = \sum_{x \in \mathcal{X}} p(x)$$

2. When the sample space $\mathcal{X}$ is **continuous**:
   - Example: Stopping time for a car with $\mathcal{X} = [3,12]$
   - **Question:** What is the probability that the stopping time is *exactly* 3.14159?

$$P(A) = \int_A p(x)dx$$

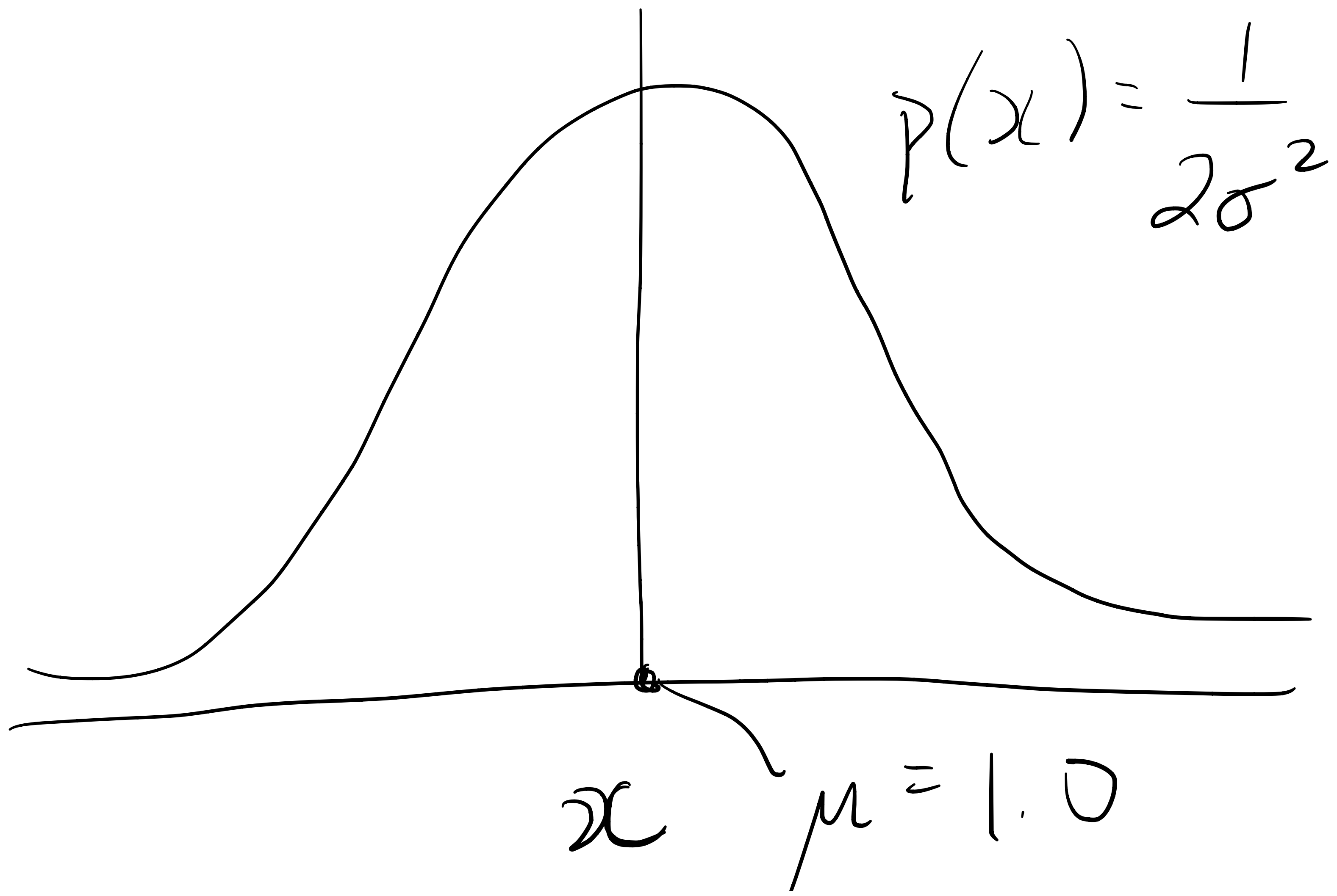
$$P(X \in \{3.14159\}) = \int_{3.14159}^{3.14159} p(x)dx = 0$$

   - More reasonable: Probability that stopping time is between 3 to 3.5.

Example comparing integration and summation

Imagine we have a Gaussian distribution

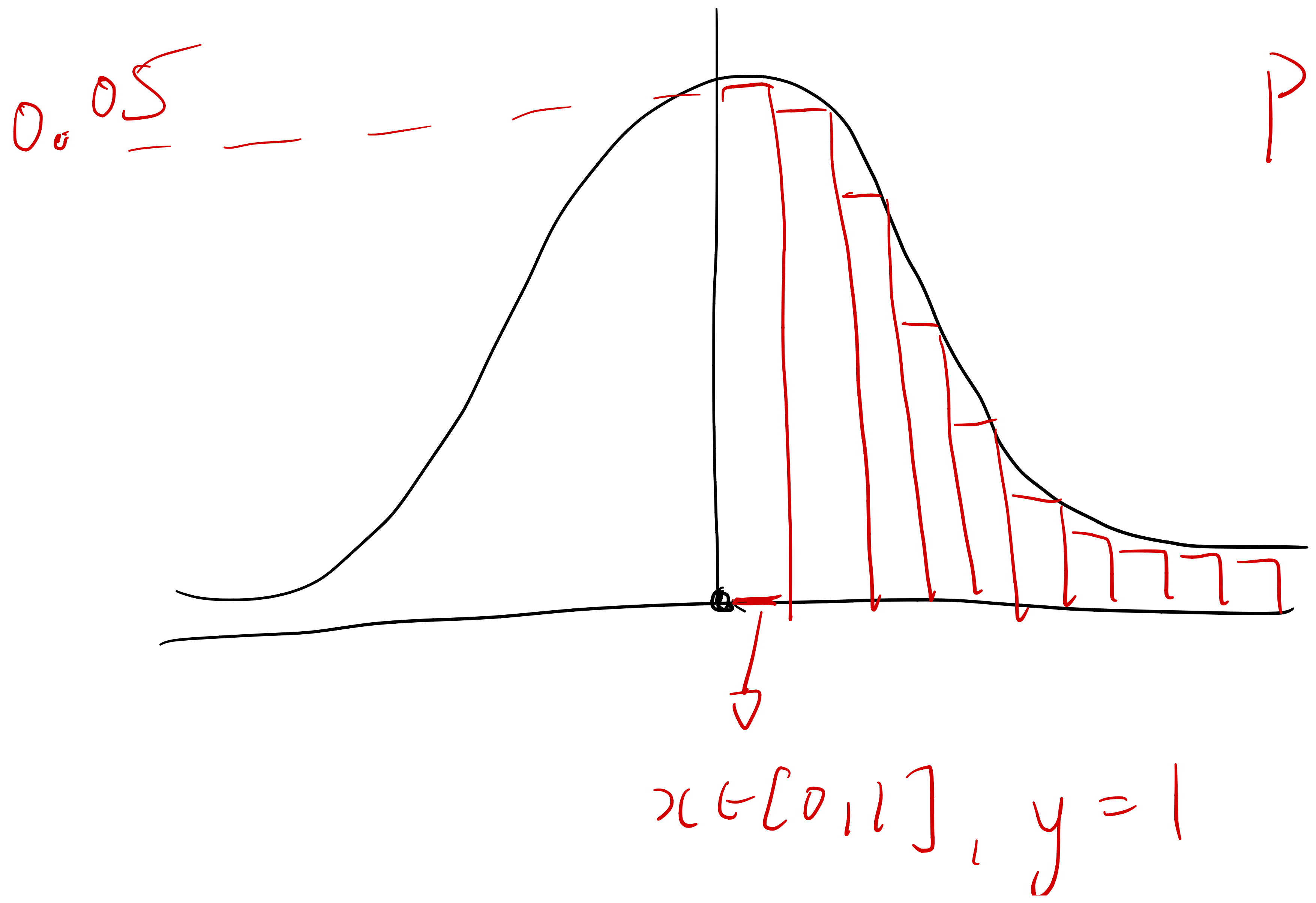
$$p(x) = \frac{1}{2\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



Example comparing integration and summation (cont)

Let's pretend we discretized to get a PMF

$y = i$ for $x \in (i-1, i]$

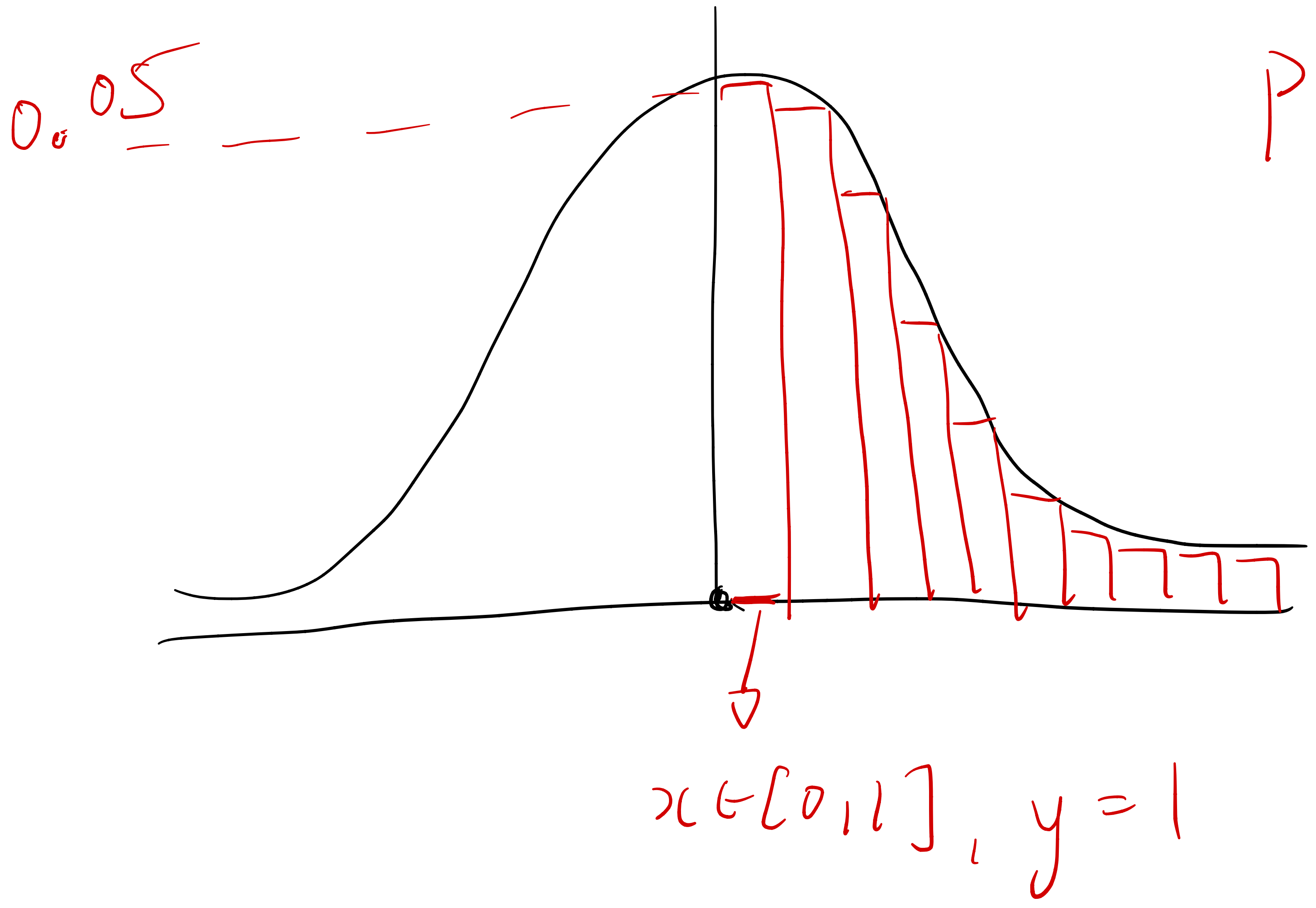


$p(y=1) = 0.05$

Example comparing integration and summation (cont)

Let's pretend we discretized to get a PMF

$y = i$ for $x \in (i-1, i]$



$p(y=1) = 0.05$

When we ask

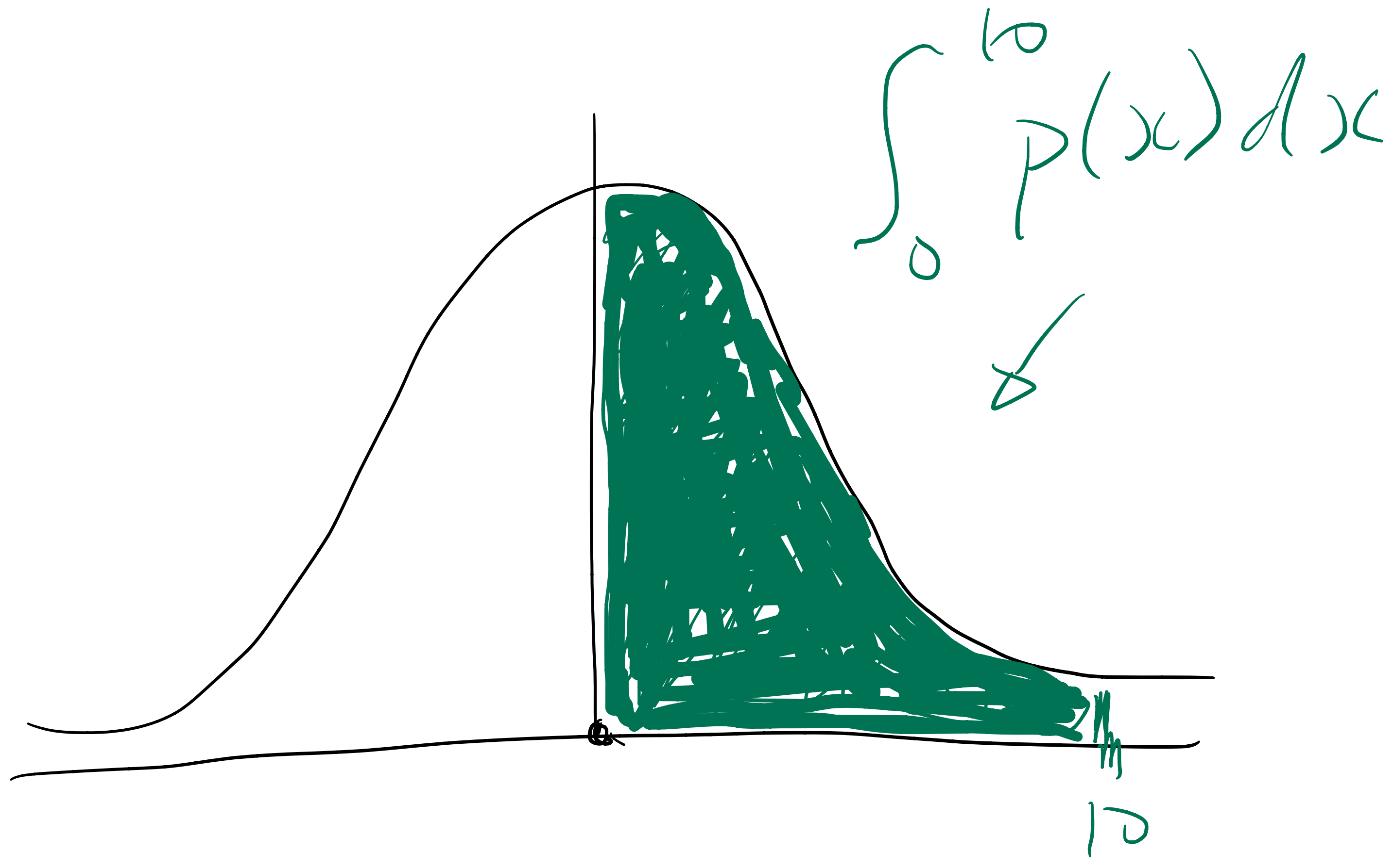
$$Pr(X \in [0, 10]) = \int_0^{10} p(x)\,dx$$

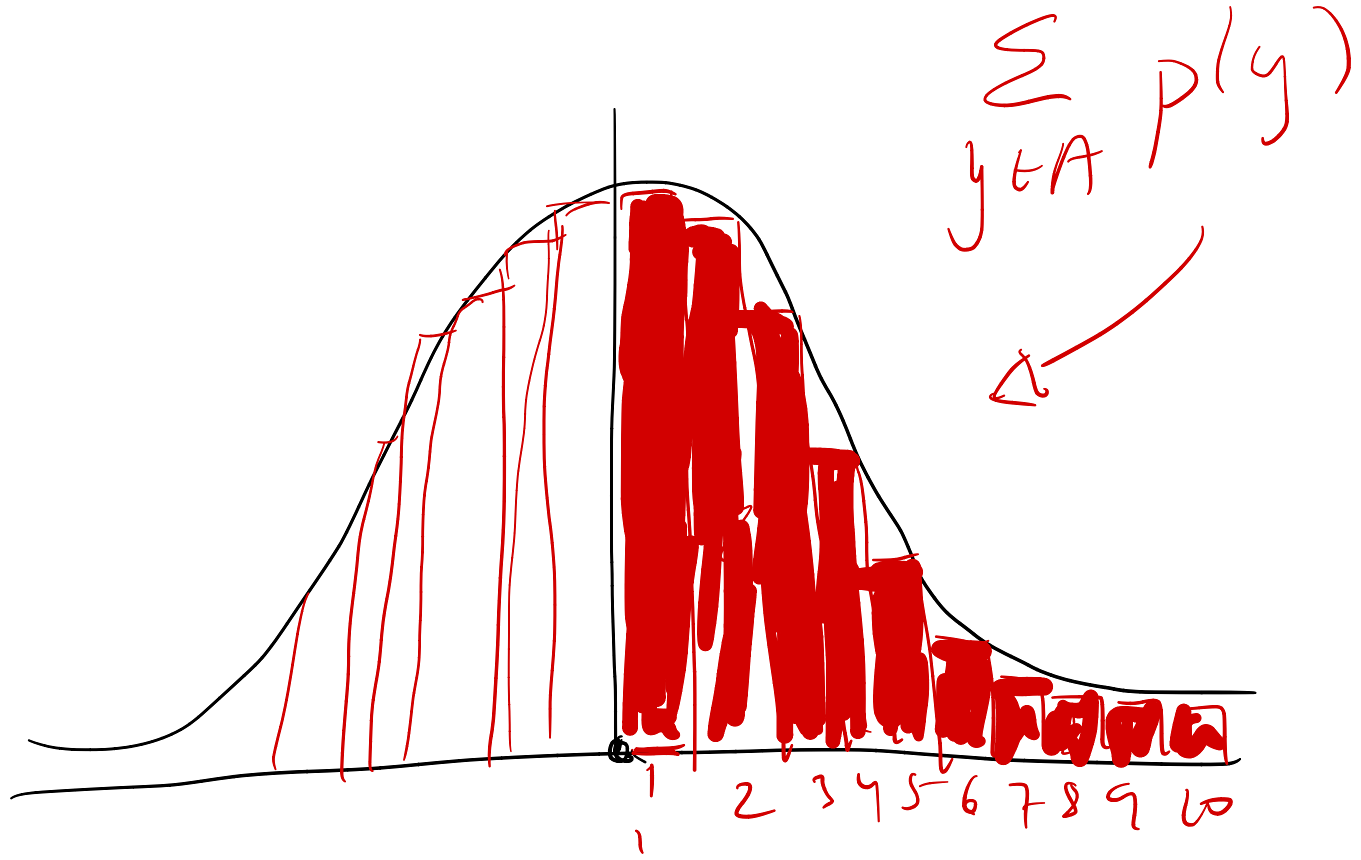
Similar to

$$Pr(Y \in \underbrace{\{1, 2, 3, \ldots, 10\}}_{A}) = \sum_{y \in A} p(y)$$

Example comparing integration and summation (cont)

Both reflect density or mass in a region.





Note: technically the red rectangles should go a bit above the Gaussian line, if we really did discretize.
My drawing is not perfect here.

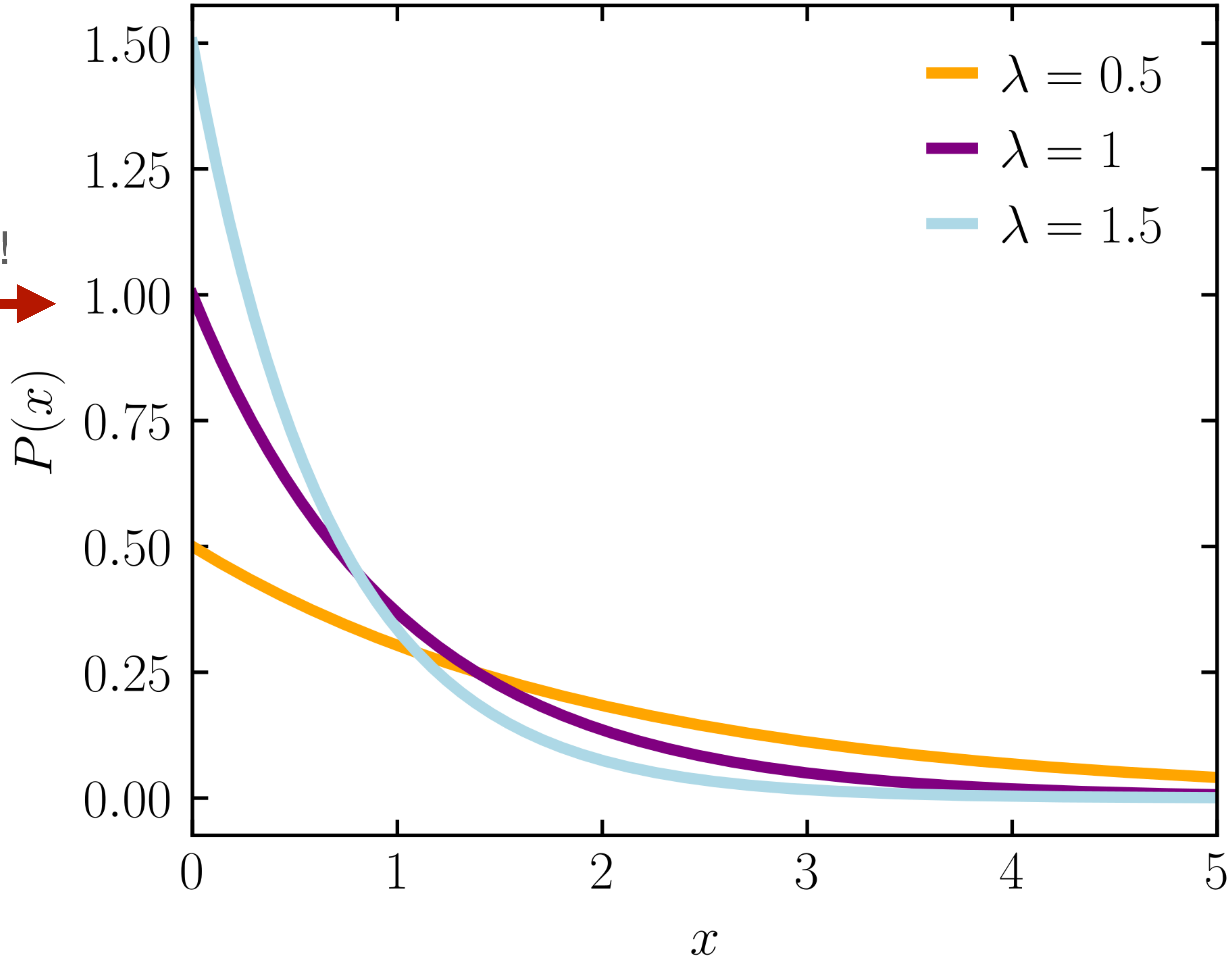
Useful PDFs: Exponential

An **exponential distribution** is a distribution over the positive reals. It has one parameter $\lambda > 0$.

$\mathcal{X} = \mathbb{R}^+$

$p(x) = \lambda \exp(-\lambda x)$

1 is here!



Why can the density be above 1?

Consider an interval event $A = [x, x + \Delta x]$, for small Δx.

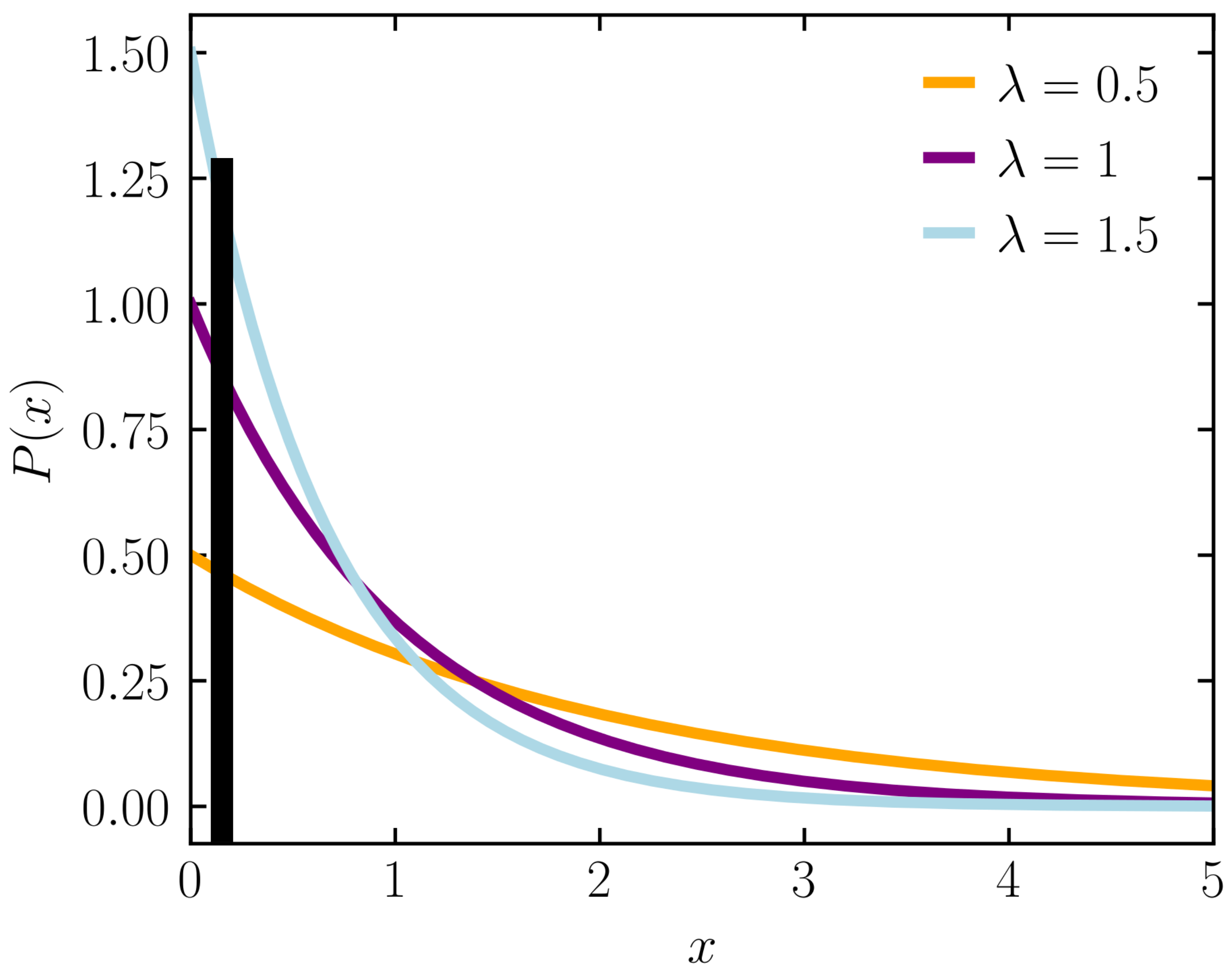
$$P(A) = \int_x^{x+\Delta x} p(x)\, dx$$

e.g., $x = 0.1, \quad \Delta x = 0.01 \quad p(x) = 1.5 \exp(-1.5x), \; p(0.1) \approx 1.3$

$$\approx p(x)\Delta x$$

$$p(X \in [0.1, 0.11]) \approx 1.3 \times 0.01 = 0.013$$

- $p(x)$ can be big, because Δx can be very small
    - In particular, $p(x)$ can be bigger than 1
- But $P(A)$ **must** be less than or equal to 1



Summary

- Probabilities are a means of **quantifying uncertainty**
- A probability distribution is defined on a measurable space consisting of a **sample space** and an **event space**.
- **Discrete** sample spaces (and random variables) are defined in terms of **probability mass functions** (PMFs)
- **Continuous** sample spaces (and random variables) are defined in terms of **probability density functions** (PDFs)
- **Random variables** let us reason about probabilistic questions with convenient boolean expressions

Exercise

- Imagine I asked you to tell me the probability that my birthday is on February 10 (day 41) or July 9 (day 190). Assume every year has 365 days (no leap years).
  - What is the outcome space and what is the event for this question?
  - Would we use a PMF or PDF to model these probabilities?
- Imagine I asked you to tell me the probability that the Uber would be here in between 3-5 minutes. Assume Uber never takes longer than 1 hour.
  - What is the outcome space and what is the event for this question?
  - Would we use a PMF or PDF to model these probabilities?

Answers

- Imagine I asked you to tell me the probability that my birthday is on February 10 (day 41) or July 9 (day 190). Assume every year has 365 days (no leap years).
  - What is the outcome space and what is the event for this question? **Outcome space is {1, 2, …., 365} and this event is {41, 190}**
  - Would we use a PMF or PDF to model these probabilities? **PMF**
- Imagine I asked you to tell me the probability that the Uber would be here in between 3-5 minutes. Assume Uber never takes longer than 60 minutes.
  - What is the outcome space and what is the event for this question? **Outcome space is [0, 60] and event is [3,5]**
  - Would we use a PMF or PDF to model these probabilities? **PDF**